Analysis of the nonlinear dynamics of flexible two-layer beams, with account for their stratification

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Abstract: The mathematical model of a two-layer beam set taking into account the geometric nonlinearity on the basis of well-known kinematic hypotheses of the first (Euler-Bernoulli), the second (Timoshenko) and the third approximations (Reddy-Pelekh-Sheremetyev) is presented. We show also that it is possible to construct mathematical models, when each layer is described by its own hypothesis. Three problems are addressed. Problem 1 - each of beams is described by the first approximation of the kinematic hypothesis. Problem 2 - each of the beams is described by the second approximation. Problem 3 - each of the beams is described by the third approximation. An external spatially distributed harmonic load acts on the beam package. The lamination of the beam structure along the entire length can occur. The stratification will lead to a change in the design algorithm scheme. In order to get reliable results, it is necessary to solve the problem taking into account two types of nonlinearity, i.e. geometric and constructive ones. A lot of attention in the work is paid to the reliability of the results. The methods for calculating such systems as systems with an infinite number of degrees of freedom have been developed. The convergence of the finite differences method is studied and the convergence of Runge-Kutta type methods is investigated. Furthermore, the value of the largest Lyapunov exponent employing three different algorithms (Wolf, Kantz and Rosenstein) is estimated.

1. Introduction

The aim of the work is to study the nonlinear dynamics of the contact interaction of flexible two-layer beams with a small clearance, described by the kinematic hypotheses of the first, second and third approximation. The influence of the inertial component and the theory of the curved normal on the nonlinear dynamics of the beam structure are investigated. It is necessary to determine what is new accounting for deformations associated with transverse forces, and allowance for the inertia of rotation in the non-linear dynamics of beam structures. On one of the beams (beam 1) is subjected to a distributed alternating load, the second beam, comes into motion due to contact with the beam 1.

Owing to complexity of equations governing the non-linear dynamics of two geometrically non-linear beams with a contact interaction, it is impossible to find an exact analytical solution. In general, the problem can be solved using numerical methods. However, the problem regarding reliability of the obtained results is generated [1]. In many cases errors introduced by numerical computations are identified with chaotic vibrations. Consequently, it is extremely important to determine the truth of the chaotic vibrations that arise during the contact interaction of the beams. It is known that the fundamental characteristics of chaos is associated with sensitivity to the initial conditions.

In this paper, we refer to Gulick [2] definition of chaos. Owing to the definition of chaos given by Gulick, chaotic orbits exist if there is either essential sensitivity to the initial conditions or at least one of the Lyapunov exponents is positive in each point of the considered chaotic domain.

As initial conditions, we will assume: the kinematic hypotheses, the boundary and initial conditions, the number of intervals for integrating beams in the finite difference method, methods for solving the Cauchy problem in the form of Runge-Kutta methods, time step for solving dynamics problems.

To reduce the infinite-dimensional problem to the Cauchy problem, we used the finite-difference method with approximation $O(c^2)$. The parameters of this method and the method itself are remain constant throughout the work.

Well known are the theories of the beams bending, such as the Euler-Bernoulli theory [3] (first approximation), Timoshenko theory [4] (second approximation), Reddy-Pelekh-Sheremetyev theory [5, 6] (third approximation).

In the scientific literature, one can find numerous works devoted to investigation of Euler-Bernoulli [7], Timoshenko [8], Reddy-Pelekh-Sheremetyev [9] beams. But there are no papers devoted to the study of nonlinear dynamics and the contact interaction of beams.

2. Formulation of the problem

The considered structure composed of two beams occupies a 2D space within the R^2 space with the rectangular system of coordinates given in the following way: a reference line, further called the middle line, is fixed in the beam 1, the axis OX is directed z=0 from the left to the right of the middle line, and the axis OZ is directed downwards. In the given system of coordinates, the space Ω is defined in the following way (see Fig. 1): $\Omega = \{x \in [0, a]; -h \leq z \leq h_k + 3h\}, 0 \leq t \leq \infty$.

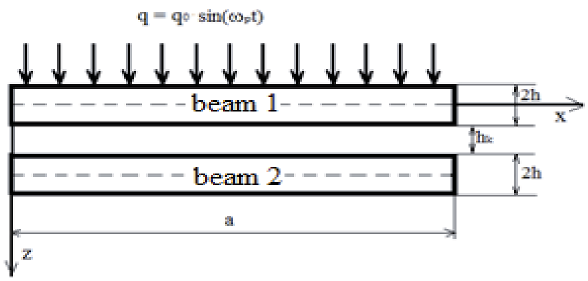


Figure 1. The settlement scheme

Equations of beams motion, as well as the boundary and initial conditions, are obtained from the Hamilton-Ostrogradskiy principle.

The contact pressure is estimated within the Kantor model [10]. The geometric non-linearity is taken in the von Kármán form. The beams are isotropic, elastic, and obey Hook's law. The longitudinal dimensions of beams are larger than their transverse dimensions and the beams have the unit thickness. There is a gap between the beams - h_k. The clearance is less than 0.2h, where h is the height of the beam, i.e. we consider small gaps between the beams.

The Cauchy problem is also solved numerically, and hence solutions essentially depend on both the chosen method and the time step integration. Therefore, in order to achieve reliable results, the Cauchy problem is solved using the Runge-Kutta of the 4th (rk4) and the 2nd (rk2) orders [11], the Runge-Kutta-Fehlberg of the 4th order (rkf45) [12, 13], the Cash-Karp of the 4th order (rkck) [14], the Runge-Kutta-Prince-Dormand of the 8th order (rk8pd) [15] as well as the implicit Runge-Kutta methods of the 2nd (rk2imp) and the 4th (rk4imp) orders.

The spectrum of the Lyapunov exponents has been estimated using three methods based on Kantz [16], Wolf [17] and Rosenstein [18] algorithms. This will ensure the reliability of the obtained numerical results.

3. Mathematical models of the contact interaction of beams described by hypotheses of the first, second and third approximations

The displacements of an arbitrary point of a beam, in the framework of the third-approximation hypothesis, are written as follows:

$$u^z = u + \underline{\underline{z\gamma_x}} + \underline{z^2u^{\mathrm{T}} + z^3\gamma^{\mathrm{T}}}\,;\; w^z = w, \tag{1}$$

where γ_x - is the transverse shear function, u^T, γ^T – are the unknown functions, w – is the deflection. We receive system of nonlinear partial differential equations for two Reddy-Pelekh-Sheremetyev beams in the displacements taking into account energy dissipation:

$$\begin{cases} \frac{1}{\lambda^2 63}\left[\frac{4}{5}\frac{\partial^3\gamma_{xi}}{\partial x^3} - \frac{1}{4}\frac{\partial^4 w_i}{\partial x^4}\right] + k^2\frac{G_{13}}{E_1}\left[\frac{\partial\gamma_{xi}}{\partial x} + \frac{\partial^2 w_i}{\partial x^2}\right] + \\ \frac{1}{\lambda^2}\left[L_3(w_i,u_i) + L_1(w_i,u_i) + \frac{3}{2}L_2(w_i,w_i)\right] - \\ -(-1)^i K(w_1 - w_2 - h_k)\Psi + q_i(t) - \frac{\partial^2 w_i}{\partial t^2} - \varepsilon_1\frac{\partial w_i}{\partial t} = 0, \\ \frac{\partial^2 u_i}{\partial x^2} + L_4(w_i,w_i) - \frac{\partial^2 u_i}{\partial t^2} = 0, \\ \frac{204}{315}\frac{\partial^2\gamma_{xi}}{\partial x^2} - \frac{48}{315}\frac{\partial^3 w_i}{\partial x^3} - 12\lambda^2k^2\frac{G_{13}}{E_1}\left[\gamma_{xi} + \frac{\partial w_i}{\partial x}\right] - \frac{\partial^2\gamma_{xi}}{\partial t^2} = 0, i = 1,2, \end{cases} \tag{2}$$

where: $i = 1,2$ - are the sequence number of beams, $L_1(w_i,u_i) = \frac{\partial^2 w_i}{\partial x^2}\frac{\partial u_i}{\partial x}$, $L_2(w_i,w_i) = \frac{\partial^2 w_i}{\partial x^2}\left(\frac{\partial w_i}{\partial x}\right)^2$, $L_3(w_i,u_i) = \frac{\partial w_i}{\partial x}\frac{\partial^2 u_i}{\partial x^2}$, $L_4(w_i,w_i) = \frac{\partial w_i}{\partial x}\frac{\partial^2 w_i}{\partial x^2}$ - are the non-linear operators, γ_{xi} - is the transverse shear function, w_i, u_i - are the deflection and displacement functions of the beams, respectively, K – is the coefficient of transverse stiffness of the contact zone.

The equation of motion of the beam element (2) contains a fourth-order derivative, which is extremely important in proving the existence of a solution of the equation and the convergence of various methods for their solution. The hypothesis of the second approximation - the of Timoshenko hypothesis, consists in the fact that tangential displacements are distributed along the beam thickness according to a linear law, i.e. in expression (1) there remain only linear terms, the terms underlined by one line are assumed to be zero.

Equations in displacements for a structure of two beams, in a dimensionless form, where both beams are described by a second approximation model:

$$\begin{cases} \frac{1}{3}\left[\frac{\partial^2 w_i}{\partial x^2}+\frac{\partial \gamma_{xi}}{\partial x}\right]+\frac{1}{\lambda^2} \\ \left[L_1(w_i,u_i)+L_3(w_i,u_i)+\frac{3}{2}L_2(w_i,w_i)\right]- \\ -(-1)^i K(w_1-w_2-h_k)\Psi+q_i(t)-\frac{\partial^2 w_i}{\partial t^2}-\varepsilon_1\frac{\partial w_i}{\partial t}=0; \\ \frac{\partial^2 u_i}{\partial x^2}+L_4(w_i,w_i)-\frac{\partial^2 u_i}{\partial t^2}=0; \\ \frac{\partial^2 \gamma_{xi}}{\partial x^2}-8\lambda^2\left[\gamma_{xi}+\frac{\partial w_i}{\partial x}\right]-\frac{\partial^2 \gamma_{xi}}{\partial t^2}=0; i=1,2. \end{cases} \tag{3}$$

$L_1(w_i,u_i), L_2(w_i,w_i), L_2(w_i,u_i), L_2(w_i,w_i)$ - these are nonlinear operators analogous to those given after the system of equations (2). Differential equations derived from the Timoshenko hypothesis have the highest second partial derivative with respect to x, which sometimes makes the proof the convergence of certain methods difficult. To obtain the Euler-Bernoulli equations, we will assume that the tangential displacements u^z, w^z are distributed linearly along the thickness, and any cross-section normal to the midline before deformation remains after deformation by a straight line and normal to the midline, the height of the section does not change. In expression (1), the zero terms are underlined by one line and replace γ_x by a rotation angle $-\frac{\partial w}{\partial x}$ for the term underlined by two lines. Thus we obtain the Euler-Bernoulli equations.

Equations governing the dynamics of two Euler-Bernoulli beams with respect to displacements and taking into account frictional energy loss (dissipation) are governed by the following PDEs:

$$\begin{cases} \frac{1}{\lambda^2}\left[L_2(w_i,w_i)+L_1(u_i,w_i)-\frac{1}{12}\frac{\partial^4 w_i}{\partial x^4}\right]- \\ -(-1)^i K(w_1-w_2-h_k)\Psi+q_i(t)-\frac{\partial^2 w_i}{\partial t^2}-\varepsilon_1\frac{\partial w_i}{\partial t}=0; \\ \frac{\partial^2 u_i}{\partial x^2}+L_3(w_i,w_i)-\frac{\partial^2 u_i}{\partial t^2}=0;\ i=1,2, \end{cases} \tag{4}$$

where $L_1(u_i,w_i)=\frac{\partial^2 u_i}{\partial x^2}\frac{\partial w_i}{\partial x}+\frac{\partial u_i}{\partial x}\frac{\partial^2 w_i}{\partial x^2}$, $L_2(w_i,w_i)=\frac{3}{2}\frac{\partial^2 w_i}{\partial x^2}\left(\frac{\partial w_i}{\partial x}\right)^2$, $L_3(w_i,w_i)=\frac{\partial w_i}{\partial x}\frac{\partial^2 w_i}{\partial x^2}$ are the nonlinear operators.

In order to model the contact interaction of the beam within the Kantor model, we introduce the term $(-1)^i K(w_1-w_2-h_k)\Psi$, $i=1,2$ into the equation governing the beams, i - stands for the beam number. The function is Ψ defined by the formula $\Psi=\frac{1}{2}[1+sign(w_1-h_k-w_2)]$, i.e. if Ψ=1 the beams are in contact $w_1>w_2+h_k$, otherwise there is no contact between [10] (see Fig. 1).

By "beam 1" we understand the externally beam loaded, whereas "beam 2" stands for the unloaded beam. We must add boundary and initial conditions to the systems of differential equations (2) – (4). In equations (2) – (4) bars are omitted.

The system of governing PDEs supplemented by boundary and initial conditions is reduced to the counterpart dimensionless form using the following variables:

$$\overline{w}=\frac{w}{2h},\overline{a}=\frac{ua}{(2h)^2},\overline{x}=\frac{x}{a},\lambda=\frac{a}{2h},\overline{q}=q\frac{a^4}{(2h)^4E},\quad (5)$$

$$\overline{t}=\frac{t}{\tau},\tau=\frac{a}{c},c=\sqrt{\frac{Eg}{\gamma}},\overline{\varepsilon_1}=\varepsilon_1\frac{a}{c},\overline{\gamma_x}=\frac{\gamma_x a}{2h},$$

where: E – is the Young's modulus; g – is the gravity of Earth; γ – is the specific gravity of the beam material, *2h* - is the height, *a* - is the length of beams, respectively.

The beam 1 is subjected to the uniformly distributed transverse harmonic excitation of the following form:

$$q=q_0\sin(\omega_p t),\quad (6)$$

where q_0 stands for the amplitude and ω_p for the frequency of excitation.

The obtained system of non-linear PDEs (2) – (4) is reduced to ODEs using the FDM (Finite Difference Method) with the approximation $O(c^2)$, where *c* – is a step regarding the spatial coordinate. The Cauchy problem is solved using the Runge-Kutta methods in time.

On the basis of the described algorithms, the program package has been developed, which allows one to solve the given problem with respect to the control parameters $\{q_0,\omega_p\}$. The main attention has been paid to control and avoid the occurrence of penetration of the structural elements. As it has been already pointed out, the studied problems are strongly non-linear, and hence an important question regarding reliability of the obtained results arises. The analysis of the results was carried out using signals, Fourier power spectra, Poincaré pseudo-mappings, phase 2D and 3D portraits, wavelet spectra based on the Gauss 32 mother wavelet. The beam 1 is subjected to the uniformly distributed transverse harmonic excitation (6). Where $q_0=5000$, $\omega_p=5.1$. The beam clearance equals $h_k=0.1$.

4. Numerical results

4.1. Task 1. Both beams of the packet are described by the kinematic hypothesis of the first approximation

We must add boundary and initial conditions to the system of differential equations (4).

Boundary conditions for case when both ends of the beams are rigidly clamped:

$$w_i(0,t) = w_i(1,t) = u_i(0,t) = u_i(1,t) = \frac{\partial w_i(0,t)}{\partial x} = \frac{\partial w_i(1,t)}{\partial x} = 0, i = 1, 2. \quad (7)$$

Initial conditions:

$$w_i(x)_{|t=0} = 0, u_i(x)_{|t=0} = 0, \frac{\partial w_i(x)}{\partial t}_{|t=0} = 0, \frac{\partial u_i(x)}{\partial t}_{|t=0} = 0, i = 1,2. \quad (8)$$

A preliminary study was made of the convergence of the FDM.

We are compared the signals, the Fourier power spectra, the Poincaré pseudo-map, the Gauss 32 wavelet spectra, phase portraits (2D and 3D) for different values of integration intervals n=40; 80, 120; 160. It was found that complete coincidence of signals for both beams occurs when n=160. The results were obtained using the 8th order Runge-Kutta method in the modification of Prince-Dormand.

It should be emphasized, that earlier in [19] the convergence of the results for the chaotic state of the system was determined from the convergence of the Fourier power spectra, and convergence with respect to the signal was not required. After obtaining the convergence of the solution by the FDM, was made a comparison of solutions obtained by various methods of Runge-Kutta type. The results are completely the same for the Runge-Kutts of all orders.

The values of the highest Lyapunov exponent, calculated by the methods of Wolff, Rosenstein, and Kants, are compared depending on n. The obtained values result from the computation using the 8th-order Runge-Kutta method. It should be emphasized that different methods of computation of the Lyapunov exponent are needed to obtain reliable/true value and, consequently, reliable estimation of chaos. When using the FDM, for the number of beams partitions n=40, 80, 120; 160 for any mentioned method, the Lyapunov exponents coincide with an accuracy up to the third decimal digit. Furthermore, all of the largest Lyapunov exponents (LLEs) are positive. In what follows we define how the LLEs depend on the method employed for solution of the Cauchy problem. For this purpose, we have computed the Lyapunov exponents using the Wolf, Kantz and Rosenstein algorithms for n=160. On the contrary to the dynamic characteristics, we have not observed full coincidence of the results. However, the difference between the minimum and the maximum of the exponents computed for different Runge-Kutta methods using the Wolf algorithm for the beam 1 is about 0.07, whereas the same done by the Rosenstein and Kantz methods yields the difference of 0.008. Convergence up to the second decimal digit has been observed for each of the computational methods of the LEs computation. It should be noted that all values of the LLE, independently of the employed method of the solution of the Cauchy problem, the beam partition number, and the employed LLE method, are positive.

Based on the analysis of dynamic indicators and the values of LLEs, it can be concluded that the oscillations of the investigated beam structure are chaotic and the revealed chaos is true.

4.2. Task 2. Both beams of the packet are described by the kinematic hypothesis of the second approximation

Let us consider the case when both beams are described by a second-approximation model (the Timoshenko model). We must add boundary (9) and initial (10) conditions to the system of differential equations (3). Both ends of the beams are rigidly clamped:

$$w_i(0,t) = w_i(1,t) = u_i(0,t) = u_i(1,t) = \gamma_{xi}(0,t) = \gamma_{xi}(1,t) = 0, i = 1,2. \quad (9)$$

Initial conditions:

$$w_i(x)_{|t=0} = 0, u_i(x)_{|t=0} = 0, u_i(x)_{|t=0} = 0, \frac{\partial w_i(x)}{\partial t}_{|t=0} = 0, \quad (10)$$

$$\frac{\partial u_i(x)}{\partial t}_{|t=0} = 0, \frac{\partial \gamma_{xi}(x)}{\partial t}_{|t=0} = 0, i = 1,2.$$

Here the bars under the dimensionless parameters are omitted for simplicity.

A preliminary study was made of the convergence of the FDM depending on the $n = 40; 80; 120; 240;\ 360; 400$.

For n = 40; 80; 120 the deflection is very different from the deflection counted with a large n. Starting with n = 240, deflections for the first beam coincide. For the second beam, the convergence by the number of divisions of the segment is much worse and begins with n = 360. The error between the signals calculated at n = 360 and n = 400 is 3%, but the signals coincide in shape over the entire time interval. The results were obtained using the 8th order Runge-Kutta method in the modification of Prince-Dormand.

The difference in signals does not exceed 2% for beam 2 for different methods of the Runge-Kutta class at the central point of the beams, at the same time. The greatest difference is between the methods of the second and eighth order.

In addition to the convergence of the signals in the center of the beams, let us check the convergence of the solution along the length of the beam. For this purpose, the plots of the deflection of beams were compared at the same instant of time t=500 at n=40; 80; 120; 240; 360; 400. It was revealed that the shape of the median line deflection along the length of the beam completely coincides for beam 1 at n = 240; 360; 400, and for beam 2 with n = 360; 400.

To make a decision on the reliability of the obtained results and the validity of chaotic oscillations, it is necessary to achieve convergence in the spectra of beam power, wavelet spectra, 2D and 3D phase portraits, and the Poincaré section. All the dynamic indices coincided for both beams at n = 400.

Based on the above analysis, we will take it for further research, which will ensure the maximum convergence of the results of beam 1 and beam 2. To solve the Cauchy problem, it is necessary to use the Runge-Kutta methods of the 8th order.

Based on the above analysis, we will take n=400 for further research, which will ensure the maximum convergence of the results of beam 1 and beam 2. To solve the Cauchy problem, it is necessary to use the 8th order Runge-Kutta methods. Let us investigate the convergence of the values of the LLEs, calculated by three different methods-Wolff, Kantz, and Rosenstein. In Table 1 we give the values of the LLEs for different n and for different methods for solving the Cauchy problem.

Table 1. The LLEs calculated for signals obtained in solving the Cauchy problem by various methods of Runge-Kutta type with n=400.

The Runge-Kutta methods	Beam 1			Beam 2		
	Wolff	Rosenstein	Kantz	Wolff	Rosenstein	Kantz
Rk8pd	0,01658	0,05646	0,02191	0,02835	0,04617	0,02363
Rkck	0,01399	0,05528	0,04583	0,02837	0,04583	0,02156
Rkf45	0,01556	0,05035	0,02300	0,02835	0,04321	0,02128
Rk4imp	0,01468	0,04298	0,01922	0,02832	0,03887	0,02051
Rk4	0,01414	0,04228	0,01952	0,02827	0,03911	0,01812
Rk2imp	0,01464	0,03492	0,01520	0,02828	0,03428	0,01752

The LLEs for the first beam are the closest ones by the Rosenstein method, and for the second beam - according to Wolf's method. The convergence of the LE as a function of n is good in the framework of one method. Up to the second decimal place, the values obtained by the Rosenstein method at n = 400 and 360 for the beam 1 are the same. For beam 2, the values of the Lyapunov exponent obtained by the Rosenstein method differ by one-hundredth for the same n. The Kantz method also gives convergence to the second decimal point at n = 400; 360.

In all the cases, the sign of the LLEs is positive, which indicates that the vibrations of beam structure is chaotic.

4.3. Task 3. Both beams of the packet are described by the kinematic hypothesis of the third approximation

We must add boundary (11) and initial (12) conditions to the system of differential equations (2). Both ends of the beams are rigidly clamped:

$$w_i(0,t) = w_i(1,t) = u_i(0,t) = u_i(1,t) = \gamma_{xi}(0,t) = \gamma_{xi}(1,t) = 0,$$

$$\frac{\partial w_i(0,t)}{\partial t} = \frac{\partial w_i(1,t)}{\partial t} = 0;\ \frac{\partial u_i(0,t)}{\partial t} = \frac{\partial u_i(1,t)}{\partial t} = 0; \quad (11)$$

$$\frac{16}{5}\frac{\partial^2 \gamma_{xi}(0,t)}{\partial x^2} - \frac{\partial^3 w_i(0,t)}{\partial x^2} = 0;\ \frac{16}{5}\frac{\partial^2 \gamma_{xi}(1,t)}{\partial x^2} - \frac{\partial^3 w_i(1,t)}{\partial x^2} = 0;$$

$$\frac{136}{315}\frac{\partial \gamma_{xi}(0,t)}{\partial x} - 0.038\frac{\partial^2 w_i(0,t)}{\partial x^2} = 0;\ \frac{136}{315}\frac{\partial \gamma_{xi}(1,t)}{\partial x} - 0.038\frac{\partial^2 w_i(1,t)}{\partial x^2} = 0,\ i=1, 2.$$

Initial conditions:

$$w_i(x)_{|t=0} = 0, u_i(x)_{|t=0} = 0, u_i(x)_{|t=0} = 0, \quad (12)$$

$$\frac{\partial w_i(x)}{\partial t}_{|t=0} = 0, \frac{\partial u_i(x)}{\partial t}_{|t=0} = 0, \frac{\partial \gamma_{xi}(x)}{\partial t}_{|t=0} = 0, i = 1,2.$$

Analogously to the problems described above, the convergence of the results is investigated depending on the number of intervals for the partition of the beam n = 40; 80; 120; 240; 360; 400; 440 and on the method for solving the Cauchy problem. The conclusion about the convergence of the results was made on the basis of the analysis of signals, power spectra, Poincaré pseudo-mappings, phase portraits.

It was found that the coincidence of signals for beam 1 occurs at n = 360, for beam 2 at n = 400. The convergence for different methods of the Runge-Kutta class for the beam 1 is complete, and for beam 2 there are differences between the methods of the 2nd and 4th order from the 8th order method.

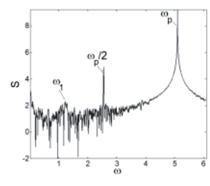
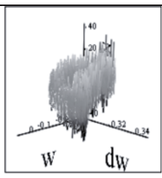
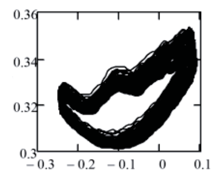
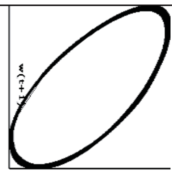
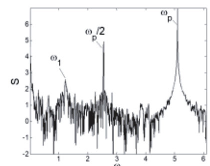
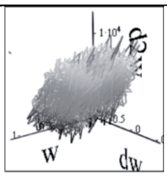
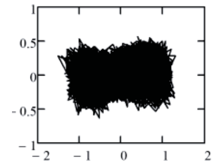
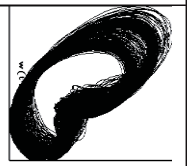
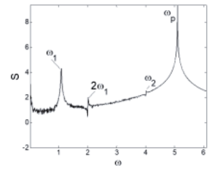
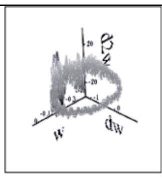
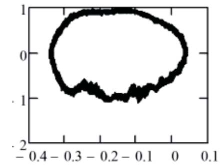
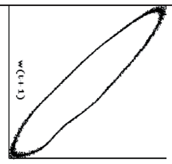
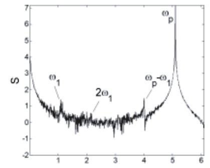
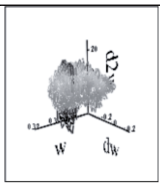
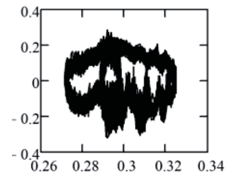
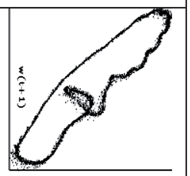
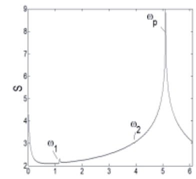
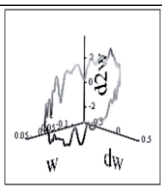
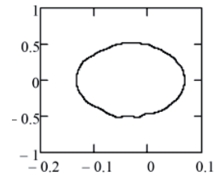
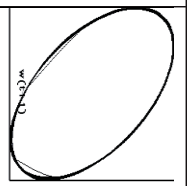
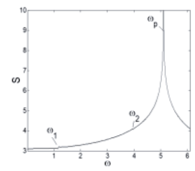
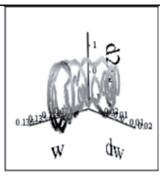
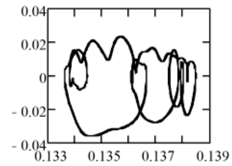
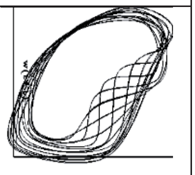
Thus, to solve the problem of the contact interaction of two beams described by the kinematic hypothesis of the third approximation, when using the FDM of the second order, the required number of partitions along the spatial coordinate must not be less than 400. As the method for solving the Cauchy problem, the Runge-Kutta method 8-th order in the Prince Dormand modification. For each considered problem, the values of the LLEs were calculated from three different algorithms. All the LLEs are positive, which allows us to speak of the truth of the chaotic oscillations of the studied beam structure.

5. Comparative analysis of the tasks 1-3

In Table 2 we give the main dynamic indicators for each of the described problems. The results obtained by the 8th order Runge-Kutta method in Prince-Dormand modification are presented.

For task 1, the vibrations of the beams are three-frequency, and the frequencies are linearly dependent on the frequencies $\omega_p = 5.1$. The phase portraits and pseudo-Poincare mappings for beams 1 and 2 have significant differences between themselves. For task 2, the Fourier power spectrum of beam 1 has four frequencies with a slight noisy at low frequencies. For the same problem, the power spectrum of beam 2 reflects a greater number of noise frequencies, but the fundamental frequencies are the same as for beam 1. All frequencies are linearly dependent. The greatest number and amplitude of noise frequencies is observed in the case of problem 3. In contrast to problems 1 and 2, on the power spectra of both beams there is the frequency of the first bifurcation $\omega_p/2$. The remaining dynamic characteristics are well correlated with the Fourier power spectra.

Table 2. Dynamic characteristics of beams vibrations.

	beam	Fourier power spectra	3D phase portraits $w(\dot{w}; \ddot{w})$	2D phase portraits $w(\dot{w})$	Poincaré pseudo-mappings
n=400; 440, task 3	1				
	2				
n=360; 400, task 2	1				
	2				
n=120; 160, task 1	1				
	2				

6. Conclusions

The continuous mechanical system cannot be truncated to the system with a finite number of degrees of freedom, but the problem is, indeed, of an infinite dimension.

We have detected, the occurrence of the phase synchronization of beams vibrations for the investigated system, among others. In addition, all frequencies exhibited by beam vibrations are in resonance relation with the excitation frequency.

Regardless of the methods for solving the problem and the hypotheses used at the modeling stage, it can be concluded that the vibrations of the two-layer beam structure are chaotic. The magnitude of the amplitude of the beam vibrations for all problems are of the same order.

It was found that with an increase in the number of partitions, the beam vibrations are regularized.

Acknowledgements

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