Mathematical models of two parametric pendulums with modulated length

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Abstract: Dynamics of a parametric pendulum excited by a wave-modulated discrete function of its length is investigated both analytically and with the use of computer simulations. An existence results of almost periodic sequences of ordinary differential equations with linear boundary value conditions are observed. Behavior of an exemplary oscillator subjected to both an almost-periodic step elongation and forcing, analogously tends to almost-periodic motions. Finally, conditions for that synchronization as well as numerical trajectories on phase planes and Poincaré sections are presented.

1. Introduction

In this study two nonlinear dynamical system of one and three degrees of freedom with a wave-modulated length and a variable-length spring pendulum [3,4,7] and a vibrating suspension [1] are mathematically derived.

Parametric excitation of a rigid planar pendulum caused by a square-wave modulation of its length is investigated in [2] both analytically and with the use of computer simulations. The threshold and other characteristics of parametric resonance are found. The role of non-linear properties of the pendulum in restricting the resonant swinging is emphasized. The boundaries of parametric instability are determined as functions of the modulation depth and the quality factor. Stationary oscillations at these boundaries and at the threshold conditions are investigated. The feedback providing active optimal control of pumping and damping is analyzed. Phase locking between the drive and the pendulum at large amplitudes and the phenomenon of parametric autoresonance are discussed.

The existence of the resonance phenomena both external and internal occurs in vibrating structures as an increased amplitude of vibrations. In general, from the engineering point of view this type of grazing behavior is usually unwanted also in solid bodies. Appearance of resonance generate greater complexity of a mechanical system behavior. In this paper, the study is performed to create the simulation and investigation for better understating of resonance phenomena of a periodically forced slider-spring pendulum mechanical system of three degrees of freedom.

Pendulum can be excited parametrically by a given vertical motion of its suspension point. In the frame of reference associated with the pivot, such forcing of the pendulum is equivalent to periodic modulation of the gravitational field [2]. This apparently simple physical system exhibits a surprisingly vast variety of possible regular and chaotic motions. Many contributions are devoted to investigation of the pendulum with vertically oscillating pivot: see, for instance [8]. A widely known curiosity in the behavior of an ordinary rigid planar pendulum whose pivot is forced to oscillate along the vertical line is the dynamic stabilization of its inverted position, occurring for the precise intervals of the driving amplitude and frequency.

The pendulum may be suspended to the flexible element. In this system the autoparametric excitation may occur as a result of inertial coupling. Analogous behavior happens when the mass is attached to the pendulum type elastic oscillator, and then, it is possible to observe autoparametric nonlinear coupling between the angle of the pendulum and elongation of the spring. All of such cases depend on the set of parameters for the investigating system. Examples are as follows: dumping, mass ratio of components, and specification of external excitation. As a result of system specification, the resonance phenomena transferring the energy between system components or their mutual excitation can appear differently.

2. Problem description

We analyze the three-degrees-of-freedom dynamical system presented in Fig. 1.

Our system consists of an elastic pendulum with the initial length l_0, the stiffness k and the damping c. The pendulum is attached to the moving slider with the point-focused mass M. The slider moves horizontally along the x-axis. The mass m hangs down from the end of the spring. The body of mass M (slider) is subjected to the harmonic vertical excitation force $F(t) = F_0\cos\omega t$. The planar mechanical system presented above has three degrees of freedom. The generalized coordinates are assumed for the angle θ between the pendulum spring and the vertical axis z (inclination angle), the incremental elongation of the spring Δs and the horizontal displacement x of the body of mass M.

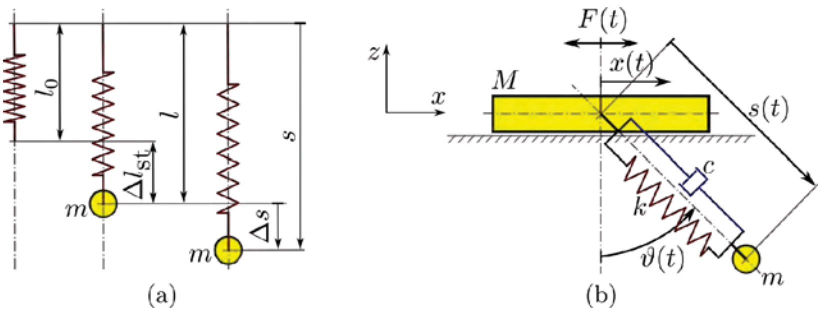


Figure 1. Dimensions of loaded (stretched or compressed) and unloaded (free) linear spring (a), a variable-length forced spring pendulum system of three degrees of freedom (b).

Any of the existing phenomena cannot be presented, examined and transferred to mathematical or engineering problem in the infinitely direct way [9]. According to this fact some assumptions allowing for a reduction of the complexity of the analyzed problem will be made. To weaken the system's complexity, but still maintaining its basic properties we have assumed:

- friction of the slider does not exists in the dynamical system;
- energy dissipated by the frictional contact of the base and the slider vibrating on it can be compensated from an external source of energy, for instance, determined by a control system;
- radial elongation of the spring pendulum exists;
- the spring is considered as massless, and its force of reaction described by Hooke's law appears when it is stretched or compressed from its free length;
- the slider has a point mass focused at the rotationally constrained end (upper) of the spring;
- excitation is caused by an external harmonic force, e.g., it can come from a magnetic field;
- mass of the spring pendulum is focused in a point at the second (lower) end of the spring;
- damping of motion is associated only with elongation of the spring of the pendulum.

We assume the almost ideal case in which the dissipation of energy by the frictional contact could be partially compensated by an external source.

3. Two mathematical models of pendulums with variable length

For the mathematical description of the dynamical system with a time-varying parameter, such as the variable length of the pendulum, the Hill or Mathieu equations are often used [10]. Nevertheless, in similar studies referring to the analyzed case, the Euler-Lagrange equation can be used.

3.1. A variable-length pendulum springily attached to the forced very weakly damped slider

The kinetic energy of the analyzed three-degrees-of-freedom system is calculated according to the sum of kinetic energies of both system bodies (see Fig. 1):

$$\begin{aligned} U(\vartheta, s, x, \dot{\vartheta}, \dot{s}, \dot{x}) &= \frac{1}{2} M\dot{x}^2 + \frac{1}{2} m\Big[\big(\dot{x} + \dot{s}\sin\vartheta + s\dot{\vartheta}\cos\vartheta\big)^2 + \big(s\dot{\vartheta}\sin\vartheta - \dot{s}\cos\vartheta\big)^2 \\ &= \frac{1}{2}\big(M+m\big)\dot{x}^2 + \frac{1}{2} m\Big[\dot{s}^2 + s^2\dot{\vartheta}^2 + 2\dot{x}\big(\dot{s}\sin\vartheta + s\dot{\vartheta}\cos\vartheta\big)\Big]. \end{aligned} \tag{1}$$

The potential energy of the analyzed mechanical system is a sum of a) the energy of the linear spring, that is accumulated after the incremental elongation Δs and the static elongation Δl_{st} (static stretching or compression by a hanging pendulum body of mass m) measured from the equilibrium free length l_0 of the spring; b) the gravitational potential energy of the body of mass m on the vertical distance $(\Delta s + l)\cos\theta$ between centers of the slider and the pendulum body, i.e.,

$$V(\vartheta, \Delta s, x) = \frac{1}{2}k(\Delta s + \Delta l_{st})^2 - mg\left(\Delta s + l\right)\cos\vartheta. \tag{2}$$

Taking into account that

$$\Delta s + \Delta l_{st} = s - l_0, \quad l = l_0 + \Delta l_{st}, \tag{3}$$

one finds

$$V(\vartheta, s, x) = \frac{1}{2}k(s - l_0)^2 - mgs\cos\vartheta. \tag{4}$$

For each component of the vector of general coordinates y_k, at independence of the assumed general coordinates, the Lagrangian $L = U - V$ satisfies the following Euler-Lagrange equation as follows:

$$\begin{gathered} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_k}\right) - \frac{\partial L}{\partial y_k} + \frac{\partial R}{\partial \dot{y}_k} = Q_k, \quad k = 1\ldots3, \\ y_k = \left[\vartheta(t), s(t), x(t)\right], \quad Q_k = \left[0, 0, F_0\cos\omega t\right], \end{gathered} \tag{5}$$

where Q_k is understood to be the reminder of the k-th generalized force when viscous damping of motion of the pendulum body in direction s is accounted for with the Rayleigh dissipation function:

$$R(\dot{\vartheta}, \dot{s}, \dot{x}) = \frac{1}{2}c\left[\frac{d\left(s - l_0\right)}{dt}\right]^2 = \frac{c\dot{s}^2}{2}. \tag{6}$$

After applying the equations (1), (4) and (6) to the Euler-Lagrange equation (5), for each generalized coordinate y_k, we get the three coupled differential equations of motion for each degree of freedom.

For the generalized coordinate θ (pendulum angle):

$$s\ddot{\vartheta} + 2\dot{s}\dot{\vartheta} + \ddot{x}\cos\vartheta + g\sin\vartheta = 0. \tag{7}$$

For the generalized coordinate s (pendulum elongation):

$$m\left(\ddot{s} + \ddot{x}\sin\vartheta - s\dot{\vartheta}^2 - g\cos\vartheta\right) + c\dot{s} + k\left(s - l_0\right) = 0. \tag{8}$$

For the generalized coordinate x (slider displacement):

$$\left(M + m\right)\ddot{x} + m\cos\vartheta\left(s\ddot{\vartheta} + 2\dot{s}\dot{\vartheta}\right) + m\sin\vartheta\left(\ddot{s} - s\dot{\vartheta}^2\right) = F_0\cos\omega t. \tag{9}$$

Equations (7)-(9) can be algebraically decoupled with respect to the second derivative, we have:

$$\ddot{\vartheta} = -\frac{1}{s}\left(2\dot{\vartheta}\dot{s} + \rho\cos\vartheta + g\sin\vartheta\right),$$
$$\ddot{s} = -\frac{c}{m}\dot{s} - \frac{k}{m}\left(s - l_0\right) + s\dot{\vartheta}^2 - \rho\sin\vartheta + g\cos\vartheta, \tag{10}$$
$$\ddot{x} = \rho,$$

where

$$\rho = M^{-1}\left(c\dot{s} + k\left(s - l_0\right)\right)\sin\vartheta + F_0 M^{-1}\cos\omega t \tag{11}$$

expresses acceleration of the slider.

The system (10) of three second order ordinary differential equations is highly non-linear due to multiplication of state variables and some trigonometric functions. It describes the continuous system dynamics that will be subject to an analysis of long term solutions that will occur far and near its resonance zones. Numerical solution of the system of equations has to be preceded by its transformation to a system of six first order differential equations, assumption of some initial conditions for the six-element state vector and also by the change of the variable $s = \Delta s + \Delta l_{st} + l_0$, so the numerical solution referred to the second degree of freedom (the state variable s) will represent an incremental elongation of the spring, i.e., Δs, about its equilibrium length l_0. The system dynamics will be investigated in the next section.

3.2. A damped mathematical pendulum with periodically modulated length

Let us consider the motion of the damped mathematical pendulum [5] with changing length $l = l(t)$ and external force $e = e(t)$ given by

$$l(t)\ddot{\phi} + c\dot{\phi} + \sin\phi = e(t). \tag{12}$$

We suppose that $l(t)$ and $e(t)$ are almost periodic step functions in the following sense: there are sequences

$$\{t_n\}_{n\in\mathbf{Z}} \subset \mathbf{R}, \quad \{l_k\}_{k\in\mathbf{Z}} \subset \mathbf{C}, \quad \{e_k\}_{k\in\mathbf{Z}} \subset \mathbf{C},$$
$$\{T_k\}_{k\in\mathbf{Z}} \subset \mathbf{C}, \quad \{w_k\}_{k\in\mathbf{Z}} \subset \mathbf{R}, \tag{13}$$

such that

$$t_n = nT + \sum_{k\in\mathbf{Z}} T_k e^{iw_k n}, \quad \forall n \in \mathbf{Z}, \quad T > 0, \quad \sum_{k\in\mathbf{Z}} \left|T_k\right| < \frac{T}{2}, \tag{14}$$

and for any $t_n < t < t_{n+1}$, we have:

$$e(t) = \sum_{k\in\mathbf{Z}} e_k e^{iw_k n}, \quad l(t) = \sum_{k\in\mathbf{Z}} l_k e^{iw_k n}, \tag{15}$$

where

$$\sum_{k\in\mathbf{Z}} \left|e_k\right| < \infty, \quad \sum_{k\in\mathbf{Z}} \left|l_k\right| < \infty. \tag{16}$$

Moreover, we suppose that $l(t)$ and $r(t)$ are step functions with almost periodic jumps. We are interested in finding conditions on $l(t)$, $e(t)$ and c that Eq. (12) has a bounded solution on **R** with the same almost periodic properties as $l(t)$ and $e(t)$. To solve this problem, a sequence of ordinary differential equations with linear boundary value conditions has to be studied. Being motivated by the approach presented in [6], considering continuous almost periodic ordinary differential equations, the boundary value problem can be solved with the use of Banach fixed point theorem together with a method of majorant functions. For a simplicity, taking into account a concrete form of Eq. (12) the solution will be found, as well as to visualize the pendulum's behavior, some numerical computations performed.

Let the difference equation be analyzed in the form:

$$x_{n+2} + ax_{n+1} + x_n = bx_n^3 + d_1 \cos n\sqrt{2} + d_2 \sin 3n, \quad n \in \mathbf{Z}, \tag{17}$$

where $a \in \mathbf{R}$, $|a| > 2$ and $b, d_1, d_2 \in \mathbf{R}$. It can be shown that if

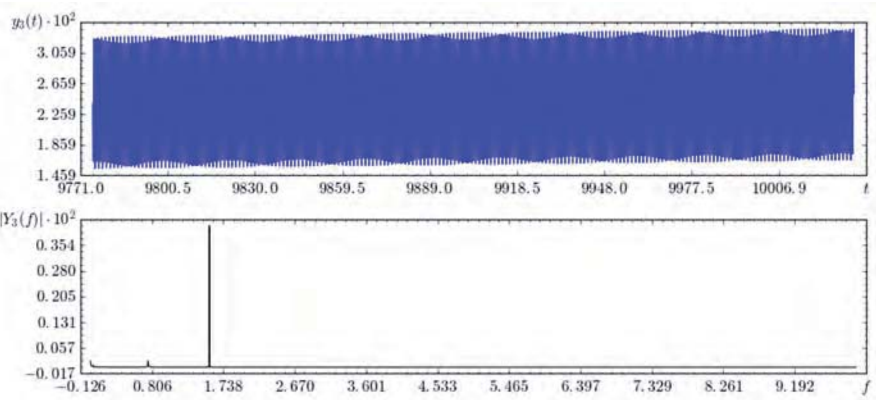
$$27\left|b\right|\left(\left|d_1\right| + \left|d_2\right|\right)^2 < 4\left(\left|a\right| - 2\right)^3 \tag{18}$$

then Eq. (18) has a solution of the form:

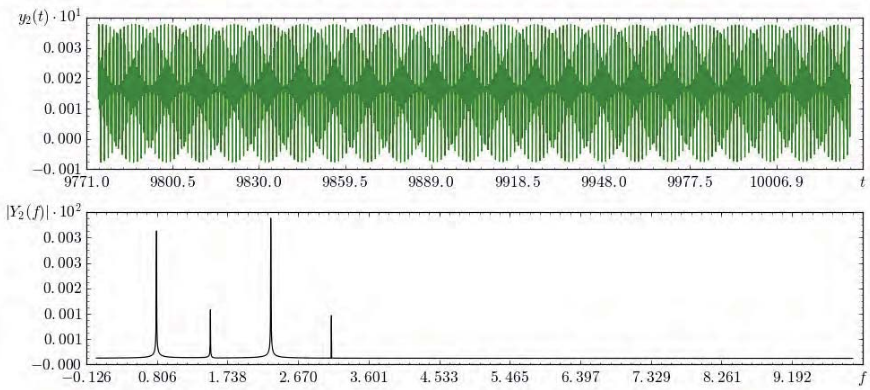
$$x_n = \sum_{k,p\in\mathbf{Z}} z_{kp} e^{i(k\sqrt{2}+3p)n}, \quad \sum_{k,p\in\mathbf{Z}} \left|z_{kp}\right| \le \frac{3}{2}\frac{\left|d_1\right| + \left|d_2\right|}{\left|a\right| - 2}. \tag{19}$$

4. Numerical computations

At this stage of our study, the dynamics of the very weakly damped pendulum is discussed. In Fig. 7, we see an interesting example of quasi-periodic oscillations of the slider-pendulum system in each degree of freedom. It is confirmed in Fig. 7b by three closed color curves on Poincaré maps. The slider oscillates quasi-periodically with the frequency $f_2 \approx 0.9707$ being synchronized with the same frequency of angular oscillations of the pendulum. Additionally, with regard to the weakly damped case and in comparison to the previous case, the elongation of the spring pendulum is much greater as well as the remaining state variables take higher maximal amplitudes of oscillations.



a)



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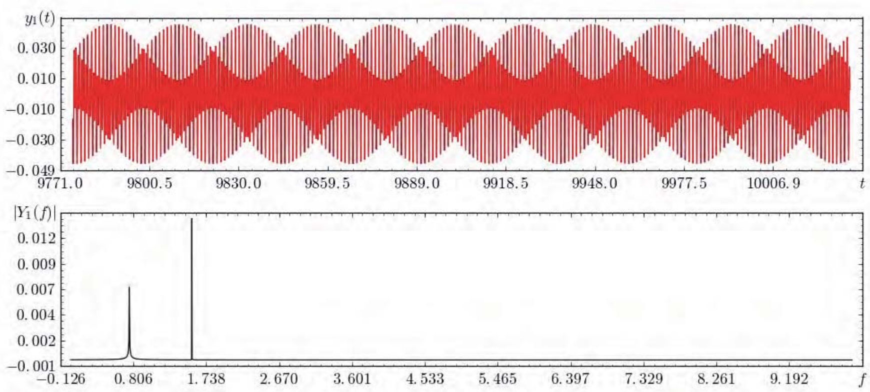
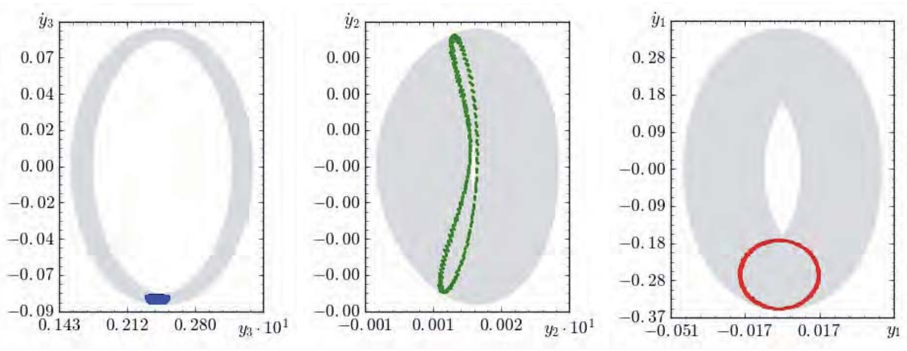


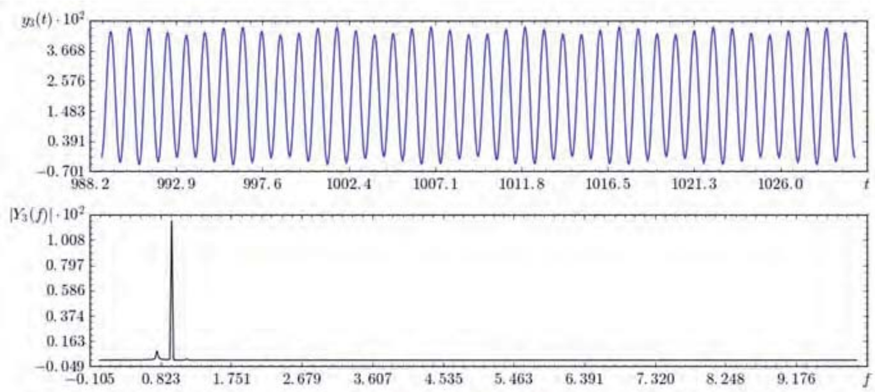
c)

Figure 2. Time histories with amplitude modulation of the length of the pendulum given by Eq. (10), phase planes (grey lines) and Poincaré maps (red, green and blue dots) for the case of weakly damped variable-length spring pendulum (see Sec. 3, c = 0.01 Ns/m). Parameters: t_0 = 9775.12, t_k = 10032.36, t_{ob} = 257.24, T = 0.6431 s, n_T = 400, ω = 9.77 rad/s.

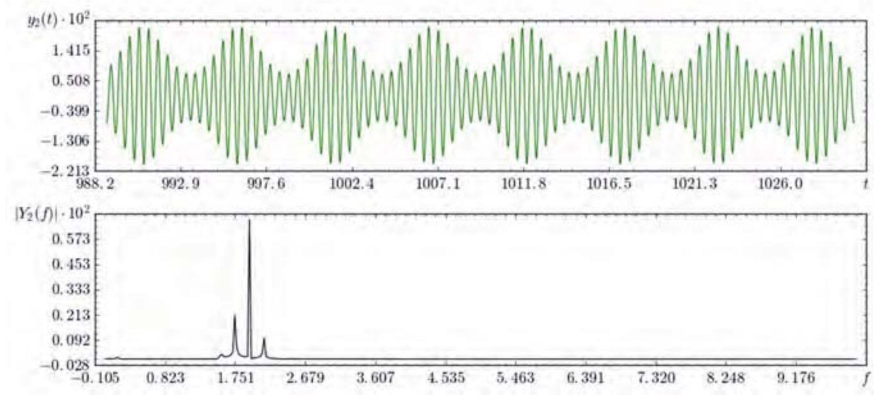


d)

Figure 2 (continued).

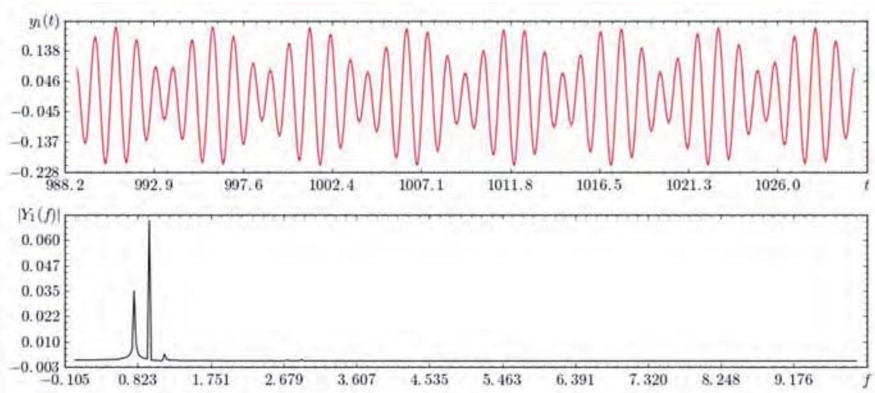


a)

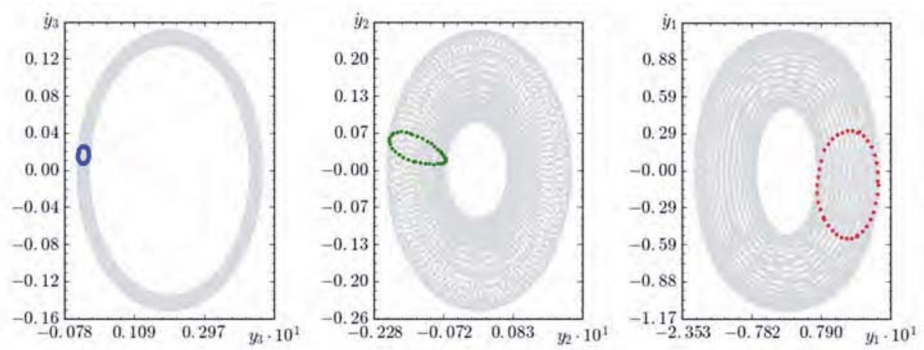


b)

Figure 3. Time histories with amplitude modulation of the length of the pendulum given by Eq. (10), phase planes (grey lines) and Poincaré maps (red, green and blue dots) for the case of weakly damped variable-length spring pendulum (see Sec. 3, $c = 0.01$ Ns/m). Parameters: $t_0 = 988.848$, $t_k = 1030.05$, $t_{ob} = 30.9015$, $T = 1.03005$ s, $n_T = 40$, $\omega = 6.1$ rad/s.



c)



d)

Figure 3 (continued).

Figures 2 and 3 represent quasi-periodic behaviour of all system bodies confirmed by closed curves on Poincaré maps. Inspecting the spectral power density plots in Fig. 2c and 3c, each mode of oscillations is associated with a slightly different frequency. It is a very characteristic dynamical behaviour since at least two different frequencies of oscillations are reported.

5. Conclusions

Two mechanical systems consisting of a variable-length pendulums were subject to a mathematical derivations and numerical computations. The two systems dynamics was investigated based on the derivation of mathematical model and the resonance plot obtained for the case of very weak damping of incremental elongation of the pendulum. The observations brought us interesting results, summarizing that the three-degrees-of-freedom mechanical system with partial dissipation of kinetic energy of motion oscillates mainly periodically and quasi-periodically. Nevertheless, the system dynamics can exhibit chaos in a close vicinity of resonance peaks of maximum amplitudes. The damped spring pendulum with a moving point of its attachment has two modes of oscillations, the pendulum angle of rotation mode and the spring incremental elongation mode. Finally, the second model of a mathematical pendulum with jumping length has been defined for proving the existence of almost periodic solutions. A mathematical analysis supported with numerical computations of the jumping discontinuity system will be taken into a deeper consideration in further works.

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