Natural oscillations of rectangular plates with holes: using Reissner's approach

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Abstract: The problem regarding the influence of holes on natural oscillations of rectangular plates has not been completely solved yet. Analytical solutions based on the traditional Rayleigh-Ritz and Bubnov-Galerkin approaches are associated with difficulties due to the approximate choice of the approximating functions for the plate deflections which should satisfy the boundary conditions. In this work, in order to study the influence of an arbitrary hole on the frequencies of a rectangular plate with an arbitrary hole, Reissner's variational principle is employed. In order to validate the proposed algorithm, a test problem is solved aimed defining the fundamental frequency of the continuous simply supported square plate. The proposed algorithm of the estimation of fundamental frequency of vibrations of the rectangular plates with a free hole possesses numerous advantages in comparison to the methods used in earlier published works. Namely, it does not introduce any limits on the dimension form and location of the hole and can be extended to study a few holes and other boundary conditions for both the plate and the hole. However, the obtained frequencies can be either larger or smaller than the exact values, and there is no any way to estimate the sign of this deviation.

The problem regarding influence of holes on the natural oscillations of rectangular plates is not completely solved yet. Analytical solutions based on the traditional Rayleigh-Ritz and Bubnov-Galerkin approaches are associated with difficulties due to the approximate choice of the approximating functions for the plate deflections which should satisfy the boundary conditions [1].

This is why the many investigations have been aimed on carrying out either experimental [2] or theoretical-experimental studies [3]. The mentioned and my other papers have been focused on study of the square plates with clamping edges, since the latter boundary conditions are easily realized experimentally. Free vibrations of simply supported square plates with centrally located circle holes have been also investigated in reference [4].

However, the results obtained there have been limited to the relatively small holes: $2r / a \leq 0.3$ (a – length of the plate side; r – hole radius).

In this work, in order to study influence of an arbitrary hole on the frequencies of rectangular plate with arbitrary hole (Fig.1) the Reissner's variational principle is employed [5]. This principle has been successfully used in reference [6] to the problems of deflections of a cantilever plate. It seems that a similar like approach can be used effectively also in the case of our problems because the fundamental difficulties of investigation of the free vibrations of the cantilever and multi-coupled plates (different boundary conditions should be satisfied on different parts of a contour) coincide.

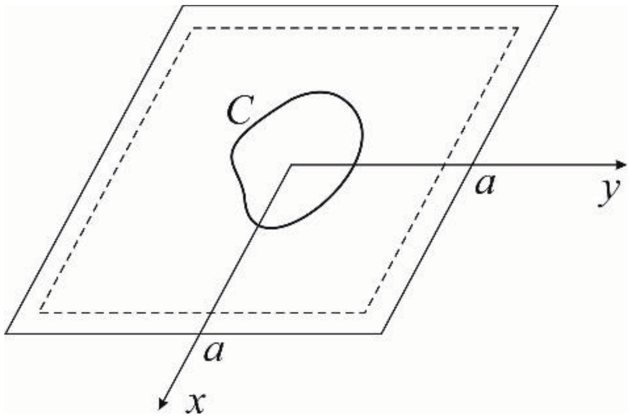

Figure 1. Investigated plate with a hole

In the case of the natural oscillations of thin plates the Reissner's variation principle takes the following form

$$\delta\left\{\iint_A\left[-M_x\frac{\partial^2 w}{\partial x^2}-M_y\frac{\partial^2 w}{\partial y^2}+2M_{xy}\frac{\partial^2 w}{\partial x\partial y}-\frac{1}{2D(1-\nu^2)}(M_x^2+M_y^2+\right.\right.$$
$$\left.\left.+2(1+\nu)M_{xy}^2)-\frac{\Omega^2\rho}{2}w^2\right]dxdy-\int_C\left[-M_n\frac{\partial w}{\partial \mathbf{n}}+\left(\frac{\partial M_n}{\partial \mathbf{n}}+2\frac{\partial M_{nt}}{\partial s}\right)w\right]ds\right\}=0, \quad (1)$$

where: w – oscillation form; $D=\dfrac{Eh^3}{12(1-\nu^2)}$; E – Young modulus; ν - Poisson's coefficient; Ω - circular frequency; ρ - mass density per unit plate surface; **n** – normal vector to the hole contour, A – area of plate, $-a\le x,y\le a$; s – coordinate along hole contour.

The variational equation (1) is equivalent to a differential equation of plate equilibrium state, physical relations of elasticity and static boundary conditions on the hole contour C (see Fig. 1).

Equation (1) is used for the approximate determination of the eigenfrequency of plate vibration in the following way. Deflection and moments are approximated independently through the function with a few (not defined yet) parameters in order to satisfy boundary conditions on the plate edges. Substituting those expressions into the variational equation (1) and carrying out the variational procedure the algebraic system of linear homogeneous equations with respect to the parameters is obtained. Comparing the determinant of the system of equations to zero, we find the equation for the eigenfrequency.

A success of the so far described method depends on how accurately the assumed functions for deflection and moments approximate the real/true functions while appropriately choosing the undefined parameters. Therefore, in order to improve convergence of the computational process it is reasonable to require satisfaction of the approximating the boundary conditions functions on the hole contour. Note, that in the latter case the curvilinear integral in formula (1) vanishes, and the input system of equations is essentially simplified.

The boundary conditions for the simply supported square plate with free hole (Fig.1) are taken

$$\text{for } \xi^2 = 1, \quad \omega = 0, \quad M_\xi = 0 \quad \text{and for } \eta^2 = 1, \quad \omega = 0, \quad M_\eta = 0, \tag{2}$$

whereas on the contour C they have the following forms

$$\begin{aligned} M_n &= M_\xi \cos^2\varphi + M_\eta \sin^2\varphi - 2M_{\xi\eta}\cos\varphi\sin\varphi = 0, \\ M_{nt} &= M_{\xi\eta}(\cos^2\varphi - \sin^2\varphi) + (M_\xi - M_\eta)\cos\varphi\sin\varphi = 0, \\ Q_n &= \left(\frac{\partial M_{\xi\eta}}{\partial\eta} + \frac{\partial M_\xi}{\partial\xi}\right)\cos\varphi + \left(\frac{\partial M_\eta}{\partial\eta} - \frac{\partial M_{\xi\eta}}{\partial\xi}\right)\sin\varphi = 0, \end{aligned} \tag{3}$$

where: $\xi = \dfrac{2x}{a}$; $\eta = \dfrac{2y}{a}$; φ - angle between the axis x and a normal to the hole contour C.

Here we have employed boundary conditions on contour C (3) in the form proposed by Poisson. This differs from the Kirchhoff's boundary conditions, since the taken ones can be solved only with respect to M_ξ, M_η, $M_{\xi\eta}$. In result, on the hole boundary we obtain

$$M_\xi = M_\eta = M_{\xi\eta} = 0. \tag{4}$$

Deflection and moments can be described by the following relations satisfying the boundary conditions

$$\begin{aligned} w &= \omega_1 \sum_{n=1}^{k} c_{1n}\varphi_{1n}, \qquad M_\xi = \omega_2 \sum_{n=1}^{k} c_{2n}\varphi_{2n}, \\ M_\eta &= \omega_3 \sum_{n=1}^{k} c_{3n}\varphi_{3n}, \qquad M_{\xi\eta} = \omega_4 \sum_{n=1}^{k} c_{4n}\varphi_{4n}. \end{aligned} \tag{5}$$

Here $\omega_i = 0$ on those parts of the boundary which require satisfactions of the boundary conditions (2), (3); φ_{in} - certain a priori chosen approximating functions; c_{in} - arbitrary constants. Now, employing the R - function method [7-9] to construct ω_i, it is not difficult to satisfy boundary conditions of the plate with a hole of an arbitrary form.

The described algorithm has been realized numerically to study influence of the free central circle hole localized in a square plate on its fundamental frequency of vibration. For simplicity, we have used only one approximating functions in the series (5), i.e.

$$
\begin{aligned}
&w = c_1 \cos\frac{\pi\xi}{2}\cos\frac{\pi\eta}{2}, \qquad M_{\xi} = c_2(\xi^2 - \eta^2 - R^2)\cos\frac{\pi\xi}{2}\cos\frac{\pi\eta}{2}, \\
&M_{\eta} = c_3(\xi^2 + \eta^2 - R^2)\cos\frac{\pi\xi}{2}\cos\frac{\pi\eta}{2}, \qquad M_{\xi\eta} = c_4(\xi^2 + \eta^2 - R^2)\sin\frac{\pi\xi}{2}\sin\frac{\pi\eta}{2},
\end{aligned}
\tag{6}
$$

where: $R = 2r/a$. Our results have been compared with those obtained by Hegarty [4] (Fig. 2). The largest error on amount of 16% occurs for small holes; increase of the hole dimension it decrease up to 10%.

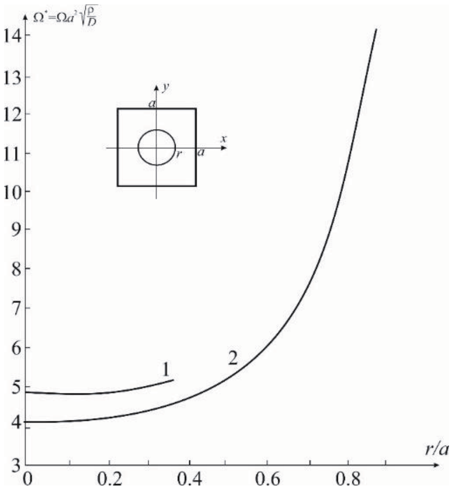

Figure 2. The fundamental frequency of the plate versus hole radius
(1 – results reported in [4]; 2 – our results)

This error has its motivation in a lack of a guarantee of the limiting transition to the continuous plate by the taken approximation of the moments (6). However, Hegarty obtained only upper estimation of the fundamental frequency, and hence the real error is lower.

In order to validate the proposed algorithm, a taste problem is solved in order to define the fundamental frequency of the continuous simply supported square plate. Its deflection and moments are approximated by the following formulas:

$$
\begin{aligned}
&\varpi = c_1 \cos\frac{N\pi\xi}{2}\cos\frac{N\pi\eta}{2}, \qquad M_{\xi} = c_2 \cos\frac{N\pi\xi}{2}\cos\frac{N\pi\eta}{2}, \\
&M_{\eta} = c_3 \cos\frac{N\pi\xi}{2}\cos\frac{N\pi\eta}{2}, \qquad M_{\xi\eta} = c_4 \sin\frac{N\pi\xi}{2}\sin\frac{N\pi\eta}{2}.
\end{aligned}
\tag{7}
$$

Note that in this case a difference comparing with the exact result achieved only 2%.

It should be emphasized that the proposed algorithms of the estimation of fundamental frequency of vibrations of the rectangular plates with free hole possesses numerous advantages in comparison to the methods used in works [1,4]. Namely, it does not introduce any limits on the dimension form and location of a hole, and can be extended to study a few of holes and other boundary conditions for the plate and hole.

There is, however, one drawback while applying the Reissner's principle to the problems of vibrations. In contrary to the Rayleigh-Ritz method, which allows to estimate frequencies located over their exact values, the Reissner's method cannot guarantee this rule [5]. In words, the obtained frequencies can be either larger or smaller than the exact values, and there is no any way to estimate a sign of this deviation.

References

[1] Kristiansen, U., Soedel, W., Fundamental of cutout square plates with clamped edges, J. Eng. Ind. 93(1) (1971) 343-345.

[2] Weaver Jr., W., Timoshenko, S.P., Young, D.H., Vibration Problems in Engineering, John Wiley & Sons, 1990.

[3] Konoplev, Yu.G., Shishkin, A.G., Free vibrations of plates and shells with holes or on point supports, in: Studies on Theory of Plates and Shells 14 (1979) Kazan, pp. 82-99 (in Russian).

[4] Hegarty, R.F., Elasto-dynamic analysis of rectangular plates with circular holes, Int. J. Solid Struc. 11(7-8) (1975) 895-906.

[5] Slivker, V., Mechanics of Structural Elements. Theory and Applications, Springer, Berlin, 2007.

[6] Plass, H.J., Gains, J.H., Newsom, C.D., Application of the Reissner's variational principle to cantilever plate deflection and vibration problems, Applied Mechanics 29(1) (1962) 127-135.

[7] Rvachev, V.L., Methods of Logic Algebra in Mathematical Physics, Kiev, Naukova Dumka, 1974 (in Russian).

[8] Shapiro, V., Semi-analytic geometry with R-Functions, Acta Numerica 16 (2007) 239-303.

[9] Kurpa, L.V., Natural vibrations of plates with holes, Soviet Applied Mechanics 15(2) (1979) 173-175.

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