Chapter 5

Discrete and Continuous Dissipative Systems

5.1 Introduction

In equations governing chaotic dynamics the dissipative properties of a system are usually represented as resistance forces proportional to a velocity. It is well known that there exist various models taking into account the dissipative properties of mechanical systems. It is known from an undergraduate course of theory of vibrations that there are linear and nonlinear resistance forces, and the nonlinear dependence can be approximated by various analytical formulas. However, when a body is cyclically deformed, a certain violation of the Hook principle may occur, which is represented by the hysteresis loop. The surface bounded by a hysteresis loop defines energy lost per one cycle of vibration in a unit material volume. It has been proved that the hysteretic loop surface, for majority of the construction materials, practically does not depend on the deformation frequency, but depends on the deformation amplitude.

We briefly revisit typical dissipation processes which are widely met while studying vibrations of the mechanical systems [Panovko (1991)].

5.2 Linear Friction

In order to study a 1-DOF system with a linear friction (damping), one may apply the following Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = -\frac{\partial \Pi}{\partial q} + Q_*, \tag{5.1}$$

where Q_* is the generalized force of a linear friction. In order to define it, we assume that in each point of the system we deal with linear dissipation

$$R_i = -\beta_i v_i, \tag{5.2}$$

where β_i is the friction coefficient. Since a general force

$$Q = \sum_{i=1}^{n} R_i \frac{\partial r_i}{\partial q}, \tag{5.3}$$

and since

$$\frac{\partial r_i}{\partial q} = \frac{\partial v_i}{\partial \dot{q}}, \tag{5.4}$$

we have

$$Q_* = -\sum_{i=1}^{n} \beta_i v_i \frac{\partial r_i}{\partial q} = -\sum_{i=1}^{n} \beta_i v_i \frac{\partial v_i}{\partial \dot{q}}. \tag{5.5}$$

From the following formula

$$v_i \frac{\partial v_i}{\partial \dot{q}} = \frac{1}{2}\frac{\partial}{\partial \dot{q}}(v_i v_i) = \frac{1}{2}\frac{\partial \nu_i^2}{\partial \dot{q}}, \tag{5.6}$$

we get

$$Q_* = -\sum_{i=1}^{n} \frac{\beta_i}{2} v_i \frac{\partial \nu_i^2}{\partial \dot{q}} = -\frac{\partial}{\partial \dot{q}} \sum_{i=1}^{n} \frac{\beta_i \nu_i^2}{2}. \tag{5.7}$$

The calculated sum

$$\Phi = \sum_{i=1}^{n} \frac{\beta_i \nu_i^2}{2}, \tag{5.8}$$

formally coincides with the kinetic energy and its associated dissipative Rayleigh function. We transform (5.8) into a more compact form

$$\Phi = \frac{1}{2} b q^2, \tag{5.9}$$

where b is the generalized coefficient of a viscosity.

Finally, we get the following formula for the generalized friction force/viscous damping force

$$Q_* = -\frac{\partial \Phi}{\partial \dot{q}} = -b\dot{q}. \tag{5.10}$$

Since the kinetic energy $T = \frac{1}{2}aq^2$ and the potential energy $\Pi = \frac{1}{2}cq^2$, the Lagrange Equation (5.1) takes the following form

$$a\frac{d^2q}{dt^2} + b\frac{dq}{dt} + cq = 0. \tag{5.11}$$

For relatively small values of the viscous damping coefficient, when $b < 2\sqrt{ac}$, the general solution to the second order differential Equation (5.11) is

$$q = e^{-ht}(C_1 \sin k_* t + C_2 \cos k_* t), \tag{5.12}$$

where

$$h = \frac{b}{2a}, \quad k_* = \sqrt{h^2 - k^2}, \tag{5.13}$$

and constants C_1 and C_2 are defined by the following initial condition

$$q(0) = q_0, \quad \frac{dq}{dt}(0) = \frac{dq_0}{dt}, \tag{5.14}$$

which means that

$$C_1 = \frac{\frac{dq_0}{dt} + hq_0}{k_*}, \quad C_2 = q_0. \tag{5.15}$$

Another solution form follows

$$q = Ae^{-ht}\sin(k_* t + \alpha), \tag{5.16}$$

where

$$A = \sqrt{\frac{\left(\frac{dq_0}{dt} + hq_0\right)^2}{h^2 - k^2} + q_0^2}, \quad \tan\alpha = \frac{q_0\sqrt{h^2 - k^2}}{\dot{q}_0 + hq_0}. \tag{5.17}$$

As it can be seen from (5.12) and (5.16), the motion can be viewed as the damped vibration with a constant frequency, but with successively decreasing amplitudes, and the full process is characterized by a monotonous amplitude dissipation.

Envelopes of the damped oscillations are described by the function

$$A = \pm A_0 e^{-ht}, \tag{5.18}$$

where A_0 stands for the initial envelope coordinate.

Frequency of the free damped vibrations is defined by the formula

$$k_* = \sqrt{h^2 - k^2} = \frac{\sqrt{b^2 - 4ac}}{2a}, \tag{5.19}$$

and time length of one cycle is

$$T_* = \frac{2\pi}{k_*} = \frac{4\pi a}{\sqrt{b^2 - 4ac}}. \tag{5.20}$$

Since the influence of the damping force on the eigenfrequency of the oscillating process is small, $k_* \approx k$, $T_* \approx T$.

Sequence of maximal deviations fits the geometric progression rule, since owing to (5.18) the ratio of two successive maximum deviations $A(t)$ and $A(t+T_*)$, separated by time interval T_* stands for a constant value equal to e^{hT_*}. Natural logarithm of that ratio is called the logarithmic decrement, which is defined as follows

$$\Lambda = hT_* = \frac{2\pi b}{\sqrt{4ac - b^2}} \approx \frac{\pi b}{\sqrt{ac}}. \tag{5.21}$$

The logarithmic decrement measures a way of damping of the free vibrations.

For essentially large values of the damping coefficient, when $b > 2\sqrt{ac}$, the general solution to (5.11) instead of (5.12), takes the following form

$$q = C_1 e^{s_1 t} + C_2 e^{s_2 t}, \tag{5.22}$$

where

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{5.23}$$

Constants of integration are expressed through the initial conditions and take the following form

$$C_1 = \frac{-C_2 e^{s_2 t} + q_0}{e^{s_1 t}}, \qquad C_2 = \frac{-s_2 q_0 + \frac{dq_0}{dt} e^{s_1 t}}{s_2 e^{s_2 t} e^{s_1 t} - s_1 e^{s_2 t}}. \tag{5.24}$$

Motion governed by (5.22) does not have vibration character: for arbitrary initial conditions, the values q and $\dot{q}$ asymptotically tend to zero.

The case when $b = 2\sqrt{ac}$ (critical damping), the solution to the differential Equation (5.11) is as follows

$$q = e^{-kt}[C_1 e^{-kt} + tC_2 e^{-kt}], \tag{5.25}$$

where $C_1 = (q_0 - t\frac{dq_0}{dt})/e^{-kt}$, $C_2 = (\frac{dq_0}{dt})/e^{-kt}$.

5.3 Nonlinear Friction

In the case when amplitudes decrease is different from that of the so far illustrated geometric progression, we deal with a nonlinear friction.

Nonlinear dependence of the friction forces versus velocity can be approximated by different analytical formulas. We assume that the generalized friction force Q_* is proportional to the nth power of velocity, whereas the power exponent $n \neq 1$ depends on the actual properties of the friction force. Thus, dependence can be cast to the following form

$$Q_* = -b|\dot{q}|^{n-1} q. \tag{5.26}$$

The governing fundamental equation takes the form

$$a\frac{\partial^2 q}{\partial t^2} + b\left|\frac{\partial q}{\partial t}\right|^{n-1}\frac{\partial q}{\partial t} + cq = 0. \tag{5.27}$$

Exact solution of the nonlinear Equation (5.27) in the form of elementary functions is not known, and hence in order to find $q = q(t)$ one may apply various types of approximating methods.

(a) Method of energy balance

Let the solution being sought be close to harmonic one and be characterized by the frequency k, corresponding to the conservative system without friction. Now, considering an arbitrary vibration cycle and beginning time measurement with time instant where the deflection achieves maximum, the motion is described by the function

$$q = A(t)\cos kt, \tag{5.28}$$

where $A(t)$ is the slowly changing function in time, i.e. $AT \ll A$, $A \ll Ak$. Then in the generalized velocity

$$q = -Ak \sin kt + A \cos kt, \tag{5.29}$$

one may omit the second term, and hence

$$q = -Ak \sin kt. \tag{5.30}$$

Owing to (5.26), the following generalized friction force is defined as

$$Q_* = -b(Ak)^n |\sin kt|^{m-1} \sin kt. \tag{5.31}$$

Work of the friction force in the considered cycle is

$$U = \int_0^T Q_* \dot{q} dt = -bk^{n+1} \int_0^T [A\,|\sin kt|\,]^{m-1} dt. \tag{5.32}$$

We may assume here approximately that in the considered period the quantity A is constant, hence

$$U = -4b(Ak)^{m+1} \int_0^{T/4} \sin^{m+1} kt dt = -4bA^{m+1}k^n \int_0^{\pi/2} \sin^{m+1}\varphi \, d\varphi. \tag{5.33}$$

The integral appearing in (5.33) will be denoted by I, and it is approximated by the Gamma function in the following way:

$$I = \int_0^{\pi/2} \sin^{m+1}\varphi \, d\varphi = \frac{2^{n-2} m^2 \Gamma\left(\frac{n}{2}\right)}{m(m+1)\Gamma(m)}. \tag{5.34}$$

Now, owing to formula (5.34), one may compute following values versus the exponent m (see Table 5.1). It is easy to find that

$$U = -4bAk^{m+1}k^{m+1} I(n). \tag{5.35}$$

Formula (5.35) presents the system energy change within the considered cycle. Since at the beginning and at the end of the considered

Table 5.1 m versus I [see (5.34)].

m	0	0.5	1.0	1.5	2.0	2.5	3.0
I	1.000	$7/8 = 0.875$	$\pi/4 = 0.785$	0.718	$2/3 = 0.667$	0.624	0.589

cycle the kinetic energy is equal to zero, the change of the full system energy is defined by the change of the potential energy, i.e. it is necessary to take into account $A(0)$ and $A(T)$. At the beginning of the cycle, we have

$$\Pi(0) = \frac{1}{2} cA^2(0), \tag{5.36}$$

whereas at the end, we have

$$\Pi(T) = \frac{1}{2} cA^2(T). \tag{5.37}$$

Consequently, the increase (negative) in the system potential energy is

$$\Delta\Pi = \frac{1}{2} c[A^2(T) - A^2(0)] = \frac{1}{2} c[A(T) + A(0)][A(T) - A(0)]. \tag{5.38}$$

After a few transformations, the following finite difference equation is obtained as

$$\Delta A = -\frac{4b(Ak)^m I(n)}{c}. \tag{5.39}$$

This equation matches the amplitude increase (negative) per one cycle with the amplitude value at the cycle beginning, i.e. it defines the shape of the upper envelope. Considering this envelope as a continuous curve governed by time function $A = A(t)$, the following approximating formula holds

$$\Delta A = T\frac{dA}{dt} = \frac{2\pi}{k}\frac{dA}{dt}. \tag{5.40}$$

Therefore, the equation of infinite differences (5.39) takes the form of the following differential equation for the envelope:

$$\frac{dA}{dt} = -\frac{2bk^{m+1} I(m)}{\pi c} A^n. \tag{5.41}$$

Integrating this equation requires considering two cases: $m = 1$ and $m \neq 1$. In the case $m = 1$ (linear damping), owing to the definition in Table 5.1 $I = \pi/4$, Eq. (5.41) takes the following form

$$\frac{dA}{dt} = -hA, \tag{5.42}$$

where

$$h = \frac{bk^2}{2c} = \frac{b}{2a}. \tag{5.43}$$

A solution to the linear Equation (5.49) is as follows

$$A = A_0 e^{-ht}, \tag{5.44}$$

where A_0 stands for the initial coordinate of the envelope. Therefore, for $m = 1$ we get the previous exact result (5.44). Although this matching holds only for the envelope (due to the difference between k and k_* the graphs would be different), it supports the idea of application of the method of energetic balance.

In the case $m \neq 1$ Equation (5.41) is nonlinear, but we may find its solution, since the variables can be separated.

$$\frac{dA}{A^m} = -\frac{2bk^{m+1}I(m)}{\pi c}dt. \tag{5.45}$$

Integrating (5.45) and taking into account the initial condition $A(0) = A_0$, the following dependence is derived

$$A = \frac{A_0}{\sqrt[m-1]{1 + \frac{2b(m-1)k^{m+1}I(m)A_0^{m-1}}{\pi c}t}}. \tag{5.46}$$

A specific form of this formula depends on m. In the case, when $m = 2$ (squared friction), formula (5.46) yields

$$A = \frac{A_0}{1 + \frac{4bk^3A_0}{3\pi c}t}, \tag{5.47}$$

which means that the envelope is described by a hyperbola. Applying solution (5.46) we may also get the envelope for another important case, when $m = 0$. Owing to (5.26), this case is associated with the value

$$Q_* = -b\frac{\dot{q}}{|\dot{q}|}, \tag{5.48}$$

defining the Coulomb friction, the volume of which does not depend on the velocity magnitude. Substituting $m = 0$ into the general solution (5.46), we get

$$A = A_0 - \frac{2bk}{\pi c}t, \tag{5.49}$$

i.e. amplitudes decrease via a linear rule and the amplitudes fit the arithmetic progression. The latter result corresponds also to the exact solution.

For $m \neq 1$ the ratio of two neighborhood largest deviations is not constant. It means that the logarithmic decrement will depend on the amplitude

$$\Lambda = \ln\frac{A_i}{A_{i+1}}, \tag{5.50}$$

where i is the number of the considered cycle. If, as it has been assumed earlier, the difference $\Delta A_i = A_{i+1} - A_i$ is small in comparison to A_i, then

$$\Lambda = \ln\frac{A_{i+1} - \Delta A_i}{A_{i+1}} = \ln\left(1 - \frac{\Delta A_i}{A_{i+1}}\right) \approx -\frac{\Delta A}{A}. \tag{5.51}$$

Substituting (6.5) into the last formula, we find the dependence of the logarithmic decrement versus the amplitude of the form

$$\Lambda = \frac{4bk^m I(m)}{c}A^{m-1}. \tag{5.52}$$

Consequently, only for $m = 1$ the logarithmic decrement does not depend on the vibration amplitude and is constant within the vibration regime. For $m = 2$, in the process of damped vibrations, the logarithmic decrement decreases simultaneously with the amplitude decrease, whereas for $m = 0$ (Coulomb friction), it increases while the amplitude decreases.

(b) Method of slowly changing amplitudes

This approximating method has been proposed by Van der Pol for a wide class of systems with weak nonlinearity, when the differential equation can be presented in the following form

$$\frac{\partial^2 q}{\partial t^2} + k_0^2 q = f\left(q, \frac{\partial q}{\partial t}\right), \tag{5.53}$$

where $f(q, \frac{\partial q}{\partial t})$ is the function containing relatively small nonlinear terms. Solution to the differential Equation (5.49) takes the following

form

$$q = A\cos(k_0 t - \varphi), \tag{5.54}$$

where it is assumed that A and φ are functions of time. Depending on the properties of the introduced functions $A(t)$ and $\varphi(t)$, formula (5.54) may fit well or wrong with the harmonic vibration of frequency k_0. For constant A and φ formula (5.54) fits the harmonic vibration in the exact manner. In the case when A and φ are "almost constant", i.e. they exhibit slow changes in time, formula (5.54) describes vibrations with slowly changing amplitude and phase. This behaviour is typical for system dynamics with a weak nonlinearity.

If we substitute formula (5.54) into the fundamental Equation (5.53), we get an equation with two unknown functions A and φ. In order to carry out properly the change of one function q by two functions A and φ, we need to add one more additional relation between them. Van der Pol proposed the following one

$$\frac{\partial A}{\partial t}\cos(k_0 t - \varphi) + A\frac{\partial \varphi}{\partial t}\sin(k_0 t - \varphi) = 0. \tag{5.55}$$

If one integrates formula (5.53) then, taking into account (5.55), the following simple formula for velocity is obtained:

$$\frac{\partial q}{\partial t} = -k_0 A\sin(k_0 t - \varphi), \tag{5.56}$$

which is similar to that of constant A and φ. Therefore, formulas for accelerations will be relatively simple and they will not contain the second-order derivative $\frac{\partial^2 A}{\partial t^2}$ and $\frac{\partial^2 \varphi}{\partial t^2}$:

$$\frac{\partial^2 q}{\partial t^2} = -\frac{\partial A}{\partial t}k_0\sin(k_0 t - \varphi) - Ak_0^2\cos(k_0 t - \varphi) + Ak_0\frac{\partial \varphi}{\partial t}\cos(k_0 t - \varphi). \tag{5.57}$$

Substituting formulas (5.54)–(5.56) into the given Equation (5.53), we get the following first-order differential equation

$$-\frac{\partial A}{\partial t}k_0\sin\psi + Ak_0\frac{\partial \varphi}{\partial t}\cos\psi = f[A\cos\psi - Ak_0\sin\psi], \tag{5.58}$$

where $\psi = k_0 t - \varphi$.

Relations (5.55) and (5.58) yield the following derivatives:

$$\begin{aligned}\frac{\partial A}{\partial t} &= -\frac{1}{k_0}f[A\cos\psi - Ak_0\sin\psi]\sin\psi,\\ \frac{\partial \varphi}{\partial t} &= \frac{1}{Ak_0}f[A\cos\psi - Ak_0\sin\psi]\cos\psi.\end{aligned} \tag{5.59}$$

Assuming that the studied system is close to the linear one, we may suspect that A and φ are not able to get relatively large increase per one cycle $2\pi/k_0$ and that derivatives $\frac{\partial A}{\partial t}$ and $\frac{\partial \varphi}{\partial t}$ are constant within a period of an arbitrary but one cycle. Therefore, although the derivatives are expressed by rather complicated nonlinear functions (5.59), we do not introduce a large error by taking their averaged values over the period $2\pi/k_0$:

$$\begin{aligned}\frac{\partial A}{\partial t} &= -\frac{1}{2\pi k_0}\int_0^{2\pi} f(A\cos\psi - Ak_0\sin\psi)\sin\psi d\psi,\\ \frac{\partial \varphi}{\partial t} &= \frac{1}{2\pi Ak_0}\int_0^{2\pi} f(A\cos\psi - Ak_0\sin\psi)\cos\psi d\psi.\end{aligned} \tag{5.60}$$

Note that carrying the integration in (5.58) we keep A as a constant value. In fact, this averaging procedure plays a key role in the described method of slowly changing amplitudes.

Equation (5.60) can be rewritten in a more compact way

$$\frac{\partial A}{\partial t} = \frac{\Phi(A)}{2\pi k_0}, \qquad \frac{\partial \varphi}{\partial t} = \frac{\Psi(A)}{2\pi Ak_0}, \tag{5.61}$$

(short Van der Pol equation), where

$$\begin{aligned}\Phi(A) &= -\int_0^{2\pi} f(A\cos\psi - Ak_0\sin\psi)\sin\psi d\psi,\\ \Psi(A) &= \int_0^{2\pi} f(A\cos\psi - Ak_0\sin\psi)\cos\psi d\psi.\end{aligned} \tag{5.62}$$

Therefore, first of all, the integrals (5.62) should be computed assuming that A is constant. Next, differential Equations (5.61) are integrated, but without an assumption of constant A.

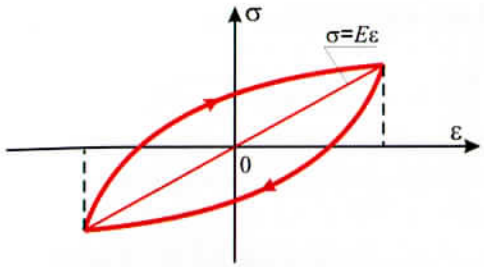

Fig. 5.1 Hysteretic loop sketch.

5.4 Hysteretic Friction

When cycling deformation of the elastic bodies takes place, even for small stresses the Hook's law is violated. In Fig. 5.1, a hysteresis loop is presented in the following coordinates: stress σ versus deformation ε. The surface located in the inside of the loop defines energy amount per one vibration cycle measured in a unit material volume. Since a distance between loop branches are usually small, getting exact hysteretic form via laboratory experiment is difficult.

It has been found that the hysteretic surface of majority of design materials does not depend on the frequency of the deformation process but rather on the deformation amplitude, which is expressed by the following formula

$$\Omega = \alpha A^{m+1}, \tag{5.63}$$

where α and m are constants defined via experiments. This dependence differs principally from the formula (5.35), where the power exponent of order $m+1$ appears. Although formula (5.35) also includes frequency k, it does not depend on the coefficient α in (5.63).

In order to define a pattern describing the damped vibration with the hysteretic friction, the equation of energetic balance will be applied. Namely, we compare the dissipated energy (it is taken with the minus sign) and energy increase governed by (5.38) per one cycle (period):

$$-\alpha A^{m+1} = cA\Delta A. \tag{5.64}$$

This approach yields the equation in the form of finite differences

$$\Delta A = -\frac{\alpha}{c} A^m, \tag{5.65}$$

which, as in (5.39), can be recast to the following differential equation

$$\frac{dA}{dt} = -\frac{\alpha k}{2\pi c} A^m. \tag{5.66}$$

Integration of Eq. (5.66) under the initial condition $A(0) = A_0$ yields

$$A = \frac{A_0}{\sqrt[m-1]{1 + \frac{\alpha(m-1)kA_0^{m-1}}{2\pi c}}} t. \tag{5.67}$$

Note that for the hysteretic friction one may get the exponential dependence (if $m = 1$), which is typical for the case of linear viscous friction. Observe also that the Coulomb friction can be treated as the particular case of the nonlinear friction (5.26), but also as the particular case of the applied fundamental formula (5.66). In both aforementioned cases, it is characterized by the value $m = 0$.

5.5 Impact Damping

In certain engineering systems, a key role of energy dissipation is played by frequent impacts instead of the so far discussed continuous action of friction forces. Let us, for simplicity, consider the case when impacts take place in time instants corresponding to the system transitions through the equilibrium position assuming that the sudden system energy loss is proportional to the system energy before the impact. In this case, the instantaneous energy loss can be measured through the system velocity before the impact

$$\Omega = b\nu^2, \tag{5.68}$$

where b is a certain constant coefficient of mass dimension.

Let us consider a half-cycle of the vibration, which begins with the largest deviation $A(0)$.

During the first quarter of the cycle, the system moves with the constant energy $\frac{1}{2}cA^2(0)$, and at the end of this quarter-cycle we have $\nu^2 = \frac{c}{A}A^2(0)$. Then, an impact takes place implying a sudden energy loss of the value of (5.68). Next, the system begins to move with the

following energy amount

$$\frac{cA^2(0)}{2} - \frac{bc}{a}A^2(0) = \frac{cA^2(0)}{2}\left(1 - \frac{2b}{a}\right), \tag{5.69}$$

which is constant within the second quarter-cycle of vibration. Therefore, at the time instant corresponding to the end of this quarter-cycle, the system potential energy is equal to that of (5.69):

$$\frac{cA^2(T/2)}{2} = \frac{cA^2(0)}{2}\left(1 - \frac{2b}{a}\right). \tag{5.70}$$

Thus, we can define a ratio of displacements at the beginning and the end of the first half-cycle as follows

$$\frac{A(0)}{A(T/2)} = \frac{1}{\sqrt{1-2b/a}}. \tag{5.71}$$

Similarly, for the second half-cycle we obtain

$$\frac{A(0)}{A(T)} = \frac{1}{\sqrt{1-2b/a}}. \tag{5.72}$$

Comparison of the largest displacements $A(0)$ and $A(T)$ yields

$$\frac{A(0)}{A(T)} = \frac{1}{\sqrt{1-2b/a}}. \tag{5.73}$$

In other words, the ratio of successive largest displacements is constant. Therefore, the envelope of the curve of the damped vibrations is characterized by the following exponent

$$A = A_0 e^{-ht}, \tag{5.74}$$

which corresponds to the logarithmic decrement of the following form

$$\Lambda = hT = \ln\frac{1}{1-2b/a}. \tag{5.75}$$

For small ratio $2b/a$, we get

$$\Lambda \approx 2b/a. \tag{5.76}$$

5.6 Damping in Continuous 1D Systems

In this section, we consider only linear continuous systems. Since internal processes of energy loss of vibrating continuous 1D systems

are very complex and require deep modeling including atomic structural patterns of a material, there is no hope, at least in the coming future, that a proper modeling of energy loss will be achieved.

5.6.1 *Free vibrations*

Let us apply a classical approach. Stress–strain relation of a linear continuous one-dimensional structural member (string, rod, beam) has the following form

$$\sigma = E\left(\varepsilon + \mu\frac{\partial\varepsilon}{\partial f}\right), \tag{5.77}$$

where E is Young's modulus, $\mu = E'/E$, E' stands for viscous damping coefficient, $\sigma(\varepsilon)$ is the stress (strain) function in an arbitrary cross-section of the analyzed mechanical object.

Considering a rod, its longitudinal force is derived using the following form

$$S\sigma = SE\left(\frac{\partial u}{\partial x} + \mu\frac{\partial^2 u}{\partial x\partial t}\right), \tag{5.78}$$

where $u(x,t)$ is the longitudinal rod displacement and S denotes the area of the rod cross-section. Dynamical equilibrium configuration of the rod infinitely small element defined by intersection of two parallel cross-sections governed by coordinates x and $x + dx$ yields the following equation

$$\rho S\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[ES\left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x\partial t}\right)\right], \tag{5.79}$$

where ρ is the rod material density. Assuming that $ES = \text{const}$ we arrive at the following PDE

$$\frac{\partial^2 u}{\partial t^2} = a^2\left[\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^3 u}{\partial^2 u\partial t}\right], \tag{5.80}$$

where $a^2 \equiv a_r^2 = E/\rho$. The same equation is derived for the case of either transversal string vibrations or rotational rod vibrations (in this case $\mu = G'/G$, where G is the shear modulus, and G' stands for the viscous damping associated with a shear processes, but with different coefficient a).

The following form of solution of (5.80) is assumed

$$u(x,t) = U(x)T(t), \tag{5.81}$$

and hence we get

$$\ddot{T}(t)U(x) = a^2[T(t)U''(x) + \mu\dot{T}(t)U''(x)] = 0, \tag{5.82}$$

or equivalently

$$\frac{\ddot{T}(t)}{\mu\dot{T}(t) + T(t)} = a^2\frac{U''(x)}{U(x)} = -\alpha^2, \tag{5.83}$$

where the constant value has been denoted by $-\alpha^2$ (here and further $' = d/dx$, $\dot{} = d/dt$). The problem has been reduced to solve the following separated two second-order differential equations

$$U''(x) + \frac{\alpha^2}{a^2}U(x) = 0, \tag{5.84}$$

$$\ddot{T}(t) + \alpha^2\mu\dot{T}(t) + \alpha^2 T(t) = 0. \tag{5.85}$$

Solutions of these equations are widely described in the books devoted to vibrations, and more recently in [Osiński (1998)] (see also Section 4.2). In order to solve the problem uniquely, we need to define boundary and initial conditions. The introduced boundary conditions allow to define the associated infinite number of *eigenfunctions* and the corresponding *eigenvalues* α_n, $n = 1, ..., \infty$ [see (5.84)]. Equation (5.85), after substitution of $T(t) = e^{rt}$, yields the following characteristic equation

$$r^2 + \alpha_n^2\mu r + \alpha_n^2 = 0, \tag{5.86}$$

and hence

$$\begin{aligned} r_{1,2} &= -\frac{\mu\alpha_n^2}{2} + \alpha_n\lambda_n, \\ \lambda_n &= \sqrt{\frac{\mu^2\alpha_n^2}{4} - 1}. \end{aligned} \tag{5.87}$$

The damped vibrations occur, when $\frac{\mu^2\alpha_n^2}{4} < 1$, and

$$T_n(t) = \left(A_n\cos\sqrt{1 - \frac{\mu^2\alpha_n^2}{4}}t + B_n\sin\sqrt{1 - \frac{\mu^2\alpha_n^2}{4}}t\right)e^{\frac{-\mu\alpha_n^2}{2}t}. \tag{5.88}$$

The upper critical damping yields the following solution

$$T_n(t) = (A_n\,\mathrm{ch}\,\lambda_n t + B_n\,\mathrm{sh}\,\lambda_n t)\,e^{\frac{-\mu\alpha_n^2}{2}t}. \tag{5.89}$$

In the case of critical damping ($\mu^2\alpha_n^2 = 4$), we get

$$T_n(t) = (A_n + B_n t)\,e^{\frac{-\mu\alpha_n^2}{2}t}. \tag{5.90}$$

It is worth mentioning that the damping depends on frequency α_n, and by increasing n the term $\alpha_n\mu$ increases. It means that even for constant μ, the successive modes of vibrations are more strongly damped [the case governed by (5.88)].

In the remaining two cases [Eqs. (5.89) and (5.90)], the system movement is aperiodic, perhaps with the occurrence of only one vibration. Free vibrations of our mechanical object are described by the following infinite series

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)U_n(x). \tag{5.91}$$

Assuming the following initial conditions

$$\begin{aligned} u(x,0) &= u_0(x), \\ v(x,0) &= v_0(x), \end{aligned} \tag{5.92}$$

we get

$$\begin{aligned} \sum_{n=1}^{\infty} T_n(0)U_n(x) &= u_0(x), \\ \sum_{n=1}^{\infty} \dot{T}_n(0)U_n(x) &= v_0(x). \end{aligned} \tag{5.93}$$

Since the eigenfunctions (modes) are orthogonal and are defined by the boundary conditions, we obtain

$$\begin{aligned} T_n(0) &= \frac{\int_0^l u_0(x)U_n(x)dx}{\int_0^l U_n^2(x)dx}, \\ \dot{T}_n(0) &= \frac{\int_0^l v_0(x)U_n(x)dx}{\int_0^l U_n^2(x)dx}, \end{aligned} \tag{5.94}$$

where l stands for the length of the structural member. Equations (5.94) define the initial conditions for Eq. (5.85). Using one

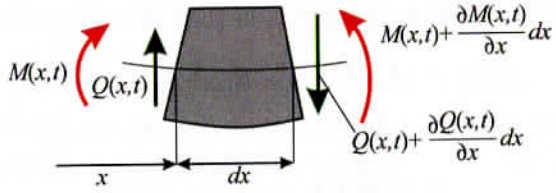

Fig. 5.2 Forces $Q(x,t)$ and torques $M(x,t)$ action on the beam element of length dx.

of Equations (5.88)–(5.90), depending on the damping, one finds the associated A_n, B_n.

Let us consider beam vibrations now. In this simple model of transversal beam vibrations $y(x,t)$ influence of transversal forces and influence of the rotational motion of beam cross-section are neglected. Using the introduced rectangular co-ordinates (axis OX is horizontal and it coincides with the beam middle line, whereas the axis OY goes down). Equation of motion of the beam element dx can be derived by taking into account Fig. 5.2.

Projection of the forces onto axis OX yields

$$\rho S dx \frac{\partial^2 y(x,t)}{\partial t^2} = -Q(x,t) + Q(x,t) + dQ(x,t), \tag{5.95}$$

where

$$Q(x,t) = \frac{\partial M(x,t)}{\partial x}. \tag{5.96}$$

In the above, E is the longitudinal elasticity modulus, I is the moment of inertia of the beam cross-section regarding the middle beam axis, and R denotes the radius of the beam curvature. In what follows, we assume that the beam material is made of a viscous and elastic material, and the stress–strain relation in the beam fibers has the following form

$$\sigma = E\left(\varepsilon + \mu\frac{\partial \varepsilon}{\partial t}\right), \tag{5.97}$$

and $\mu = E'/E$, where E' is the viscous damping coefficient.

Since

$$\frac{M}{EI} = -\left[\frac{1}{R} + \mu\frac{\partial}{\partial t}\left(\frac{y}{R}\right)\right], \tag{5.98}$$

for a small beam deflection we have

$$\frac{1}{R(x,t)} = \frac{\partial^2 y(x,t)}{\partial x^2}. \tag{5.99}$$

We finally derive the bending torque

$$M(x,t) = -EI\left(\frac{\partial^2 y(x,t)}{\partial x^2} + \mu\frac{\partial^3 y(x,t)}{\partial x^2 \partial t}\right), \tag{5.100}$$

and the transversal force

$$Q(x,t) = \frac{\partial M}{\partial x} = -EI\frac{\partial}{\partial x}\left[\frac{\partial^2 y(x,t)}{\partial x^2} + \mu\frac{\partial^3 y(x,t)}{\partial x^2 \partial t}\right], \tag{5.101}$$

and the increment of the transversal force

$$dQ = \frac{\partial Q}{\partial x}dx = -EI\frac{\partial}{\partial x^2}\left[\frac{\partial^2 y(x,t)}{\partial x^2} + \mu\frac{\partial^3 y(x,t)}{\partial x^2 \partial t}\right]dx. \tag{5.102}$$

Taking into account (5.102) in (5.95), we get

$$\frac{\partial^4 y(x,t)}{\partial x^4} + \mu\frac{\partial^5 y(x,t)}{\partial x^4 \partial t} + a_b^2\frac{\partial^2 y(x,t)}{\partial t^2} = 0, \tag{5.103}$$

where $a_b^2 = EI/\rho S$. Equation (5.103) governs free beam transversal vibrations with viscous internal damping. Assuming the solution in the form (5.81), we get

$$U^{IV}(x)\left[T(t) + \mu\dot{T}(t)\right] + a_b^2 U(x)\ddot{T}(t) = 0, \tag{5.104}$$

and hence

$$-\frac{U^{IV}(x)}{U(x)} = a_b^2\frac{\ddot{T}(t)}{T(t) + \mu\dot{T}(t)} = -k^4. \tag{5.105}$$

Finally, the problem is reduced to the study of the following two independent linear ODEs

$$U^{IV}(x) - k^4 U(x) = 0, \tag{5.106}$$

$$\ddot{T} + \mu\alpha^2\dot{T}(t) + \alpha^2 T(t) = 0, \tag{5.107}$$

where $\alpha^2 = k^4/a_b^2$. It should be emphasized that Eq. (5.106) defines the eigenfunctions (modes) and they do not depend on the internal viscous damping.

We solve linear ODE (5.106), i.e. assuming $U(x) = e^{rx}$ we obtain the following characteristic equation

$$r^4 - k^4 = 0, \tag{5.108}$$

and hence $r_1 = k$, $r_2 = -k$, $r_3 = ik$, $r_4 = -ik$. It means that a general solution of equation (5.106) is

$$U(x) = C_1 e^{kx} + C_2 e^{-kx} + C_3 e^{ikx} + C_4 e^{-ikx} \tag{5.109}$$

or equivalently

$$U(x) = A \sin kx + B \cos kx + C \text{ sh } kx + D \text{ ch } kx, \tag{5.110}$$

where four constant values A, B, C, D are defined via the boundary conditions.

We restrict our further analysis only to the case of free-free beam support. For the beam of length l the boundary conditions follow

$$y(0,t) = 0, \qquad y(l,t) = 0, \tag{5.111}$$

$$\left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=0} = 0, \qquad \left.\frac{\partial^2 y(x,t)}{\partial x^2}\right|_{x=l} = 0. \tag{5.112}$$

Substituting (5.81), (5.110) into (5.111), (5.112) yields

$$\begin{gathered} B + D = 0, \qquad -Bk^2 + Dk^2 = 0, \\ A \sin kl + B \cos kl + C \text{ sh } kl + D \text{ ch } kl = 0, \\ -Ak^2 \sin kl - Bk^2 \cos kl + Ck^2 \text{ sh } kl + Dk^2 \text{ ch } kl = 0. \end{gathered} \tag{5.113}$$

The characteristic equation is defined by

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ 0 & -k^2 & 0 & k^2 \\ \sin kl & \cos kl & \text{sh } kl & \text{ch } kl \\ -k^2 \sin kl & -k^2 \cos kl & k^2 \text{sh } kl & k^2 \text{ch } kl \end{vmatrix} = 0 \tag{5.114}$$

or equivalently

$$k^4 \text{sh } kl \sin kl = 0. \tag{5.115}$$

Since sh $kl \neq 0$ for $kl \neq 0$, we get

$$\sin kl = 0, \tag{5.116}$$

which means that

$$k_n l = n\pi, \quad n = 1, 2, \ldots. \tag{5.117}$$

Each k_n defines

$$U_n(x) = A_n \sin k_n x + B_n \cos k_n x + C_n \text{ sh } k_n x + D_n \text{ ch } k_n x. \tag{5.118}$$

Each of the infinite number of eigenfunctions should satisfy the boundary conditions. Substitution of (5.117) into algebraic Equations (5.113) yields

$$\begin{gathered} B_n = 0, \qquad D_n = 0, \\ C_n = -\frac{\sin k_n l}{\text{sh } k_n l} A_n = -\frac{\sin n\pi}{\text{sh } n\pi} A_n = 0, \end{gathered} \tag{5.119}$$

and hence

$$U_n(x) = A_n \sin k_n x = \sin n\pi \frac{x}{l}. \tag{5.120}$$

Equation (5.107) is the same as Eq. (5.86), and its solution has been studied earlier.

5.6.2 *Excited vibrations*

In this section, we consider 1D structural members (strings, rods, beams) subjected to external load action. The load $q = q(x,t)$ is continuously distributed along a structural member length per its unit length and depends on time. In the case of a longitudinally vibrating rod, the load (force) is distributed along its length continuously and in parallel to the rod axis x, whereas in the case of the rod vibrating torsionally the torque is distributed along its length. In the case of the string transversal vibrations, the load (torque) is continuously distributed along its length, and is perpendicular to the string axis.

One may show, proceeding in the way described in Section 5.6.1, that in all cases the governing equation has the following form

$$\frac{\partial^2 u(x,t)}{\partial t^2} - a^2\left[\frac{\partial^2 u(x,t)}{\partial x^2} + \mu\frac{\partial^3 u(x,t)}{\partial t \partial x^2}\right] = bq(x,t), \tag{5.121}$$

where $b = 1/(\rho S)$.

In the case of a string $a^2 = T/(\rho S)$, (T is the string tension, ρ is the string material density, S is the area of string cross-section). In the case of the rod longitudinal (torsional) vibrations $a^2 = E/\rho, b = 1/(\rho S)$ ($a^2 = G/\rho, b = 1/(\rho I_0)$), where G is the shear modulus, I_0 is the moment of inertia of the rod cross-section, and $ES = \text{const}$, $GI_0 = \text{const}$.

The studied PDE Eq. (5.121) is linear and non-homogeneous. Its solution consists of a sum of a general solution of the homogeneous PDE ($q = 0$) and a particular solution of the non-homogeneous PDE. We have already shown how to find the general solution to the homogeneous equation and how to determine constants satisfying boundary and initial conditions.

In the case of a particular solution to the non-homogeneous equation it should satisfy the boundary conditions, whereas the initial conditions follow

$$\begin{aligned} u(x,0) &= 0, \\ \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} &= 0. \end{aligned} \tag{5.122}$$

We apply the following approximation

$$q(x,t) = \sum_{n=1}^{\infty} Q_n(t)U_n(x), \tag{5.123}$$

and the solution is assumed to be of the following form

$$u(x,t) = \sum_{n=1}^{\infty} \xi_n(t)U_n(x). \tag{5.124}$$

In order to find $Q_n(t)$, we multiply both sides of Eq. (5.123) by $U_m(x)$, and compute an integral from 0 to l to get

$$\int_0^l q(x,t)U_m(x)dx = \sum_{n=1}^{\infty} Q_n(t)\int_0^l U_n(x)U_m(x)dx. \tag{5.125}$$

Since U_n and U_m are mutually orthogonal, then

$$\begin{aligned} Q_n(t) &= \frac{1}{\gamma_n^2}\int_0^l q(x,t)U_n(x)dx, \\ \gamma_n^2 &= \int_0^l U_n(x)dx. \end{aligned} \tag{5.126}$$

We substitute (5.123) and (5.124) into (5.121) and we obtain

$$\sum_{n=1}^{\infty}\ddot{\xi}_n(t)U_n(x) - \sum_{n=1}^{\infty}a^2[\xi_n(t)U_n''(x) + \mu\dot{\xi}_n(t)U_n''(x)] = b\sum_{n=1}^{\infty}Q_n(t)U_n(x), \tag{5.127}$$

which yields

$$\frac{\ddot{\xi}_n(t) - bQ_n(t)}{\xi_n(t) + \mu\dot{\xi}_n(t)} = a^2\frac{U_n''(x)}{U_n(x)} = -\omega_n^2. \tag{5.128}$$

Finally, we separate time and space dependent functions to get

$$U_n''(x) + \frac{\omega_n^2}{a^2}U_n(x) = 0, \tag{5.129}$$

$$\ddot{\xi}_n(t) + \mu\omega_n^2\dot{\xi}_n(t) + \omega_n^2\xi_n(t) = bQ_n(t). \tag{5.130}$$

Equation (5.130) governs oscillations of damped non-autonomous linear oscillators. Taking into account our earlier considerations corresponding to (5.130), we have the following solutions.

(i) Undercritical damping ($4 > \omega_n^2\mu^2$)

$$\begin{aligned} \xi_n(t) &= \frac{1}{bc_n}\int_0^t Q_n(\tau)e^{-\frac{1}{2}\omega_n^2\mu(t-\tau)}\sin c_n(t-\tau)d\tau, \\ c_n &= \omega_n\sqrt{1 - \frac{\omega_n^2\mu}{4}}. \end{aligned} \tag{5.131}$$

(ii) Supercritical damping ($4 < \omega_n^2\mu^2$)

$$\xi_n(t) = \frac{1}{bc_s}\int_0^t Q_n(\tau)e^{-\frac{1}{2}\omega_n^2\mu(t-\tau)}\,\mathrm{sh}\,c_s(t-\tau)d\tau, \tag{5.132}$$

$$c_s = \omega_n\sqrt{\frac{\omega_n^2\mu}{4}-1}.$$

(iii) Critical damping

$$\xi_n(t) = \frac{1}{b}\int_0^t Q_n(\tau)e^{-\frac{1}{2}\omega_n^2\mu(t-\tau)}(t-\tau)d\tau. \tag{5.133}$$

Finally, the equation governing transversal beam vibrations, where the beam is continuously transversally loaded by the load $q = q(x,t)$ per beam unit length, has the following form [see Eq. (5.103)]

$$\frac{\partial^4 y(x,t)}{\partial x^4} + \mu\frac{\partial^5 y(x,t)}{\partial x^4\partial t} + a_b^2\frac{\partial^2 y(x,t)}{\partial t^2} = \frac{q(x,t)}{EI}. \tag{5.134}$$

Substituting

$$y(x,t) = \sum_{n=1}^{\infty}\xi_n(t)U_n(x), \tag{5.135}$$

$$q(x,t) = \sum_{n=1}^{\infty}Q_n(t)U_n(x), \tag{5.136}$$

into (5.134) we get

$$\xi_n(t)U_n^{IV}(x) + \mu\dot{\xi}_n(t)U_n^{IV}(x) + a_b^2\ddot{\xi}_n(t)U_n(x) = Q_n(t)U_n(x) \tag{5.137}$$

or equivalently

$$-\frac{U_n^{IV}(x)}{U_n(x)} = \frac{a_b^2\ddot{\xi}_n(t) - \frac{1}{\rho F}Q_n(t)}{\xi_n(t)+\mu\dot{\xi}_n(t)} = -k_n^4. \tag{5.138}$$

Equations (5.138) yield two separated sets of ordinary differential equations

$$U_n^{IV}(x) - k_n^4U_n(x) = 0, \tag{5.139}$$

$$\ddot{\xi}_n(t) + \omega_n^2\dot{\xi}_n(t) + \omega_n^2\xi_n(t) = \frac{1}{\rho S}Q_n(t), \tag{5.140}$$

where $\omega_n^2 = k_n^4/a_b^2$, $a_b^2 = \rho E/EI$.

Solutions of the obtained Eqs. (5.139) and (5.140) have already been discussed.