

## Mathematical model for two-dimensional dry friction modified by dither

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**Abstract:** A new dynamic two-dimensional friction model is developed which is based on the bristle theory. Actually it is the Reset Integrator Model converted into a two-dimensional space. Usually two-dimensional friction models are indeed one-dimensional models which are rotated into the slip velocity direction. However, this often used approach cannot be applied to the bristle model. That is why an idea of a two-dimensional bristle is presented. Bristle's deformation is described using polar coordinates. The carried out numerical simulation of a planar oscillator has proved that the new model correctly captures the mechanism of smoothing dry friction by dither applied via perpendicular and co-linear way regarding the body velocity. Furthermore, the introduced mathematical model captures two-dimensional stick-slip behaviour. The Cartesian slip velocity components are the only inputs to the model. In addition, our proposed model allows to describe a friction anisotropy using the bristle parameters. The paper contains results of an experimental verification of the new friction model conducted on the special laboratory rig being used to investigate the two-dimensional motion in the presence of dither as well as to validate our numerical results.

### 1. Introduction

Although friction belongs to natural and common phenomena, it is still difficult to find a general mathematical model for friction force being valid in various regimes of contact dynamics of machine elements. Fundamental problem in friction modelling is discontinuity in transition from sticking to slipping phase of motion. Even in sticking phase some microscopic motion occurs, which is called pre-sliding displacement. A force needed to initiate macroscopic motion is called break-away force. It was experimentally proven, that this force changes with a rate of increase of external force applied to contacting bodies. During motion a friction force is referred as kinetic friction. There are static effects applied to kinetic friction like Stribeck or viscous friction effects. On the other hand and from dynamical point of view there exists a hysteresis effect called frictional lag which applies to kinetic friction [1].

Over the years any general theory of friction, which explains all frictional effects has not been developed yet. On the other hand numerous mathematical models were created. In general friction models can be divided into two classes: static and dynamic models. Static models are those which describe friction phenomenon only as a function of a slip velocity. This category includes classical models, the Karnopp model and Armstrong's model. In many cases internal state variables and the appropriate differential equations are used as the attributes of dynamic friction models. Examples of

dynamic models are the Dahl model, the bristle model, the reset integrator model and the LuGre model [2,3,4].

## 2. Smoothing dry friction by dither

Dither is a word for intentionally introduced vibrations or noise. In mechanical systems with friction, vibrations of one of contacting bodies can be understood as dither. Dither may influence friction characteristic essentially. In general it plays an important role since it smoothes transition from stick to slip regimes. In addition, dither can be used to quench or even eliminate a harmful stick-slip behaviour. Noticeable effect of introducing dither into mechanical system is realised via change of system's damping character. It is a well-known fact that oscillatory motion damped only with dry-friction decays with straight-line envelopes. After introducing dither, envelopes change into exponential shapes. It means that dither changes dry friction damping into viscous damping. Furthermore, this effect possesses a directional property. Namely, the motion co-linear with dither is lightly damped in comparison to the motion perpendicular to it. However, the mechanism of modification for dither perpendicular to and co-linear with the body velocity is different. A role of the crucial parameter which influences the friction damping plays an amplitude of dither velocity.

For dither co-linear with body velocity the resultant slip velocity can be described as a difference between body velocity and dither velocity. Assuming friction described by simple Coulomb model, friction force reaches the value  $\pm F_C$  depending on a slip velocity sign, where  $F_C$  denotes here the Coulomb friction force.

Dither perpendicular to body velocity actually makes friction two-dimensional. Friction force is directed opposite to resultant slip velocity. Components of friction force are proportional to components of slip velocity, i.e. dither and body velocity. It means that, the friction force component co-linear with body velocity is modulated by both body and dither velocity. The so far given brief description can be treated as an introduction to the detailed analysis on the mechanism of smoothing dry friction by dither presented in reference [5].

## 3. The new model of two-dimensional dry friction and numerical experiments

The developed friction model is a two-dimensional interpretation of the reset integrator model, presented by Haessing and Friedland in 1991 [3]. Bristle theory introduced and developed in reference [3] concerns frictions as effect of contact and deformation of irregularities of contacting surfaces. In the reset integrator model this effect is approximated by a single bristle. Strain of the bristle ( $z$ ) is the internal state variable of the model. Strain is increasing till limiting value  $z_0$  is reached, which can be interpreted as the stiction range. After reaching this value, strain is kept on a constant level, what in-

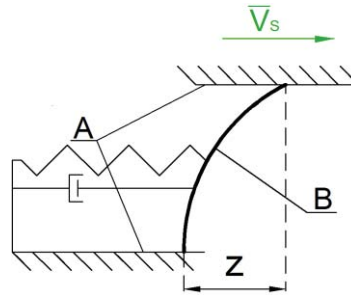
deed represents a slipping phase of the studied motion. An idea of bristle's strain and generation of friction force is presented in Fig. 3.1, where the variable  $z$  is governed by the following equation:

$$\frac{dz}{dt} = \begin{cases} 0 & \text{if } v_s > 0 \text{ and } z \geq z_0 \\ 0 & \text{if } v_s < 0 \text{ and } z \leq z_0 \\ v_s & \text{otherwise} \end{cases} \quad (3.1)$$

Friction force is divided into static and kinetic friction, whereas bristle's strain  $z$  plays the role of a switching variable:

$$F_f = \begin{cases} \sigma_0(1+a)z + \sigma_1 \frac{dz}{dt} & \text{if } |z| < z_0 \\ \sigma_0 z & \text{if } |z| \geq z_0 \end{cases} \quad (3.2)$$

In equation (3.2)  $\sigma_0$  is the bristle stiffness,  $a$  is the stiction gradient and  $\sigma_1$  stands for the damping parameter.



**Figure 3.1.** Bristle deformation: (A) contacting bodies, (B) bristle, (Z) – bristle's strain

In reference [6] two-dimensional model for investigation of stick-slip motion is presented. Components of friction force are calculated on the basis of *friction direction angle* concept, which is defined by both slip velocity and applied force. On the other hand the friction model presented in [5] allows for introducing anisotropy into friction force description. Friction force angle is chosen using the principle of maximum energy dissipation. Both models can be considered as one-dimensional models which are rotated with respect to the vector of the slip velocity. However, this approach cannot be applied to converting dynamic friction models into two-dimensional space. It is especially useless as far as interpretation of internal state variable is based on the bristle theory. Simple rotating of such a model results in loss of capturing spring-like behaviour before gross sliding occurs. It could even lead to not detecting transition between sticking and slipping phase of the studied motion. Frictional lag phenomenon is also lost. In conclusion, majority of advantages of these models is lost. That is why a different approach is highly required with respect to transformation of bristle models into a two-dimensional space.

A role of the basic parameter of every bristle model plays a stiction range. In the case of planar motion, stiction range should be described by the following planar set:

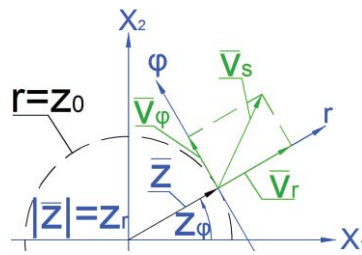
$$\Theta_z = \{ \mathbf{Z} \in \mathfrak{R}^2, \Phi_z(\mathbf{Z}) \leq 0 \}. \quad (3.3)$$

In our further investigation  $\Phi_z(Z)$  is governed by a circle formula with a radius being equal to the required pre-sliding displacement. In other words, it means that the isotropic stiction range has been assumed.

In order to obtain an appropriate physical character of the bristle model, its two-dimensional interpretation must ensure deforming bristle in two directions and hold resultant deformation during slipping phase of motion. During a planar motion the slip velocity can change its direction without obtaining value equal to zero or even without changing this value at all. It means that in the slipping phase, deformation of bristle must be hold at the same level, while its components may change. One may say that the bristle “rotates to the direction of a slip”. The so far carried out consideration leads to division of the bristle deformation into two components, i.e. rotation and strain. This division gives motivation to choose polar coordinates as the appropriate one for description of the bristle deformation:

$$\mathbf{Z} = \begin{bmatrix} z_r \\ z_\varphi \end{bmatrix}. \quad (3.4)$$

This choice has its consequences in description of the internal state variable  $Z$ . Slip velocity is distributed into two components: rotational  $v_\varphi$  and radial one  $v_r$ , which is co-linear with the deformed bristle. Mechanism of this division is shown in Fig. 3.2 Rotational component  $v_\varphi$  forces the bristle to rotate into current slip velocity direction, whereas radial component  $v_r$  is responsible for regulation of bristle’s strain.



**Figure 3.2.** Mechanism of dividing slip velocity

Resulting description of variable  $Z$  dynamic is described by the following equations:

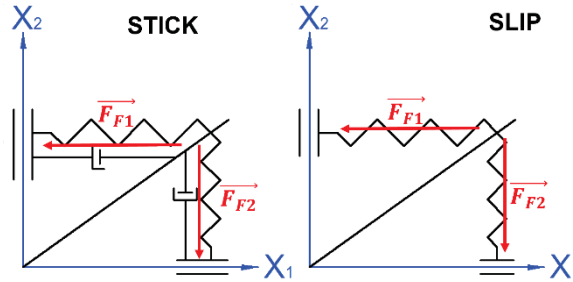
$$\frac{dz_r}{dt} = \begin{cases} 0 & \text{if } v_r > 0 \text{ and } z_r \geq z_0 \\ 0 & \text{if } v_r < 0 \text{ and } z_r \leq z_0 \\ v_r & \text{otherwise} \end{cases}, \quad (3.5)$$

$$\frac{dz_\phi}{dt} = \frac{v_\phi}{z_r}.$$

The friction force originates from the bending of the bristle. The scheme shown in Fig. 3.3 illustrates an idea of generating two-dimensional vector of the friction force. Although the bristle deformation is described by polar coordinates, the resulting friction force is given in Cartesian coordinates. In order to make our model more application oriented, the transition into polar coordinates is carried out inside the model. It means that both input and output of the model are presented in Cartesian coordinates. Like in the reset integrator model (RIM), two different components of the sticking and slipping phase while describing the friction force are introduced:

$$\mathbf{F}_F = \begin{bmatrix} F_{F1} \\ F_{F2} \end{bmatrix} = \begin{cases} \begin{bmatrix} \sigma_{01}(1+a_1)z_1 + \sigma_{11}\dot{x}_1 \\ \sigma_{02}(1+a_2)z_2 + \sigma_{12}\dot{x}_2 \end{bmatrix} & \text{if } z_r < z_0 \\ \begin{bmatrix} \sigma_{01}z_1 \\ \sigma_{01}z_2 \end{bmatrix} & \text{if } z_r \geq z_0 \end{cases} \quad (3.6)$$

Description of the friction force by equation (3.6) makes it possible to use hints from paper [3] regarding a selection of parameters of the model. Even though stiction range is assumed to be isotropic, anisotropy of friction can be introduced in description of each friction force component separately. Actually there are three parameters for both directions introduced, which can be set independently.

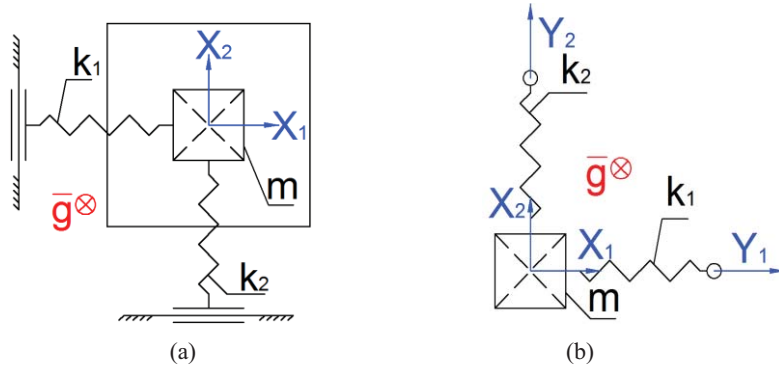


**Figure 3.3.** Idea of generating two-dimensional friction force

A few numerical experiments concerning application of the new friction model exhibiting the introduced damping parameters  $\sigma_{11}$  and  $\sigma_{12}$  as velocity dependent has been carried out. Actually they decrease with velocity increasing, owing to the following formula [1]:

$$\sigma_1(v_s) = \sigma_1 \exp \left[ - \left( \frac{v_s}{a_v} \right)^2 \right], \quad (3.7)$$

where the parameter  $a_v$  is small (for instance of order  $10^{-2}$ ).

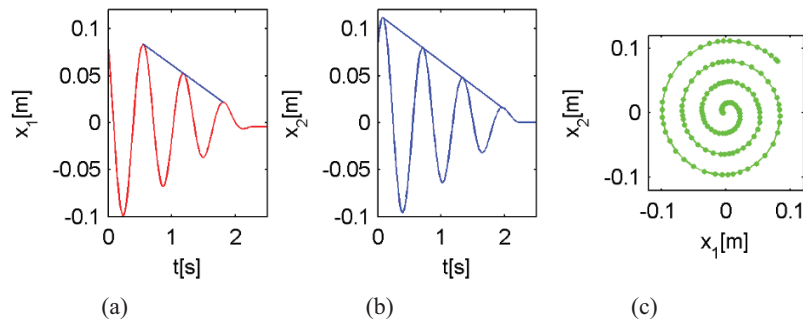


**Figure 3.4.** Planar oscillator (a) and spring-mass system for stick-slip investigation (b)

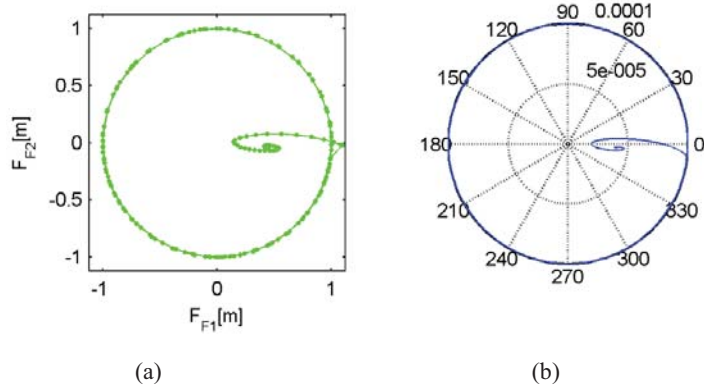
In order to validate numerical computation, a simple planar oscillator (Fig. 3.4(a)) has been studied with following fixed parameters and initial conditions:

- (i) friction parameters:  $\sigma_{01} = \sigma_{02} = 10^2[\text{N m}^{-1}]$ ,  $\sigma_{11} = \sigma_{12} = 74,15[\text{N s m}^{-1}]$ ,  $a = 0,1[-]$ ,  $z_0 = 10^{-2}[\text{m}]$ ;
- (ii) oscillator parameters:  $m = 1[\text{kg}]$ ,  $k_1 = k_2 = 100[\text{N m}^{-1}]$ ;
- (iii) initial conditions:  $x_1(t_0) = x_2(t_0) = 0,08[\text{m}]$ ,  $v_1(t_0) = -0,8[\text{m s}^{-1}]$ ,  
 $v_2(t_0) = 0,8[\text{m s}^{-1}]$ ,  $z_r(t_0) = 10^{-4}[\text{m}]$ ,  $z_\theta(t_0) = 135[^\circ]$ .

Displacement of the body in two, perpendicular directions is shown in Fig. 3.5. Observe that both  $x_1$  and  $x_2$  displacements are decaying with straight-line envelopes, what is characteristic feature for motion damped by dry friction. Fig. 3.6 shows polar diagram of radial and rotational components of the internal state variable  $z$ . Actually, this diagram shows trajectory of bristle deformation during the simulated motion. As can be seen, components of two-dimensional friction force are proportional to bristle's deformation, except the place where sticking phase begins, what has to be expected.

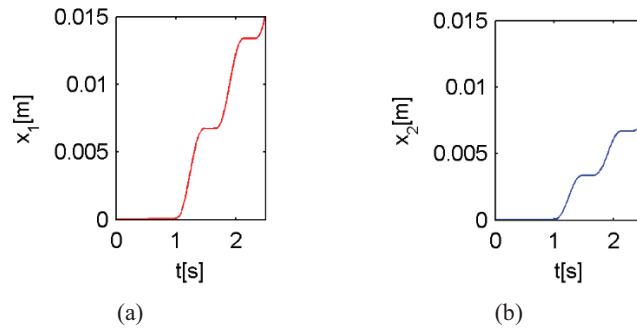


**Figure 3.5.** Simulation results: displacements  $x_1(t)$  (a) and  $x_2(t)$  (b) versus time and trajectory (c)

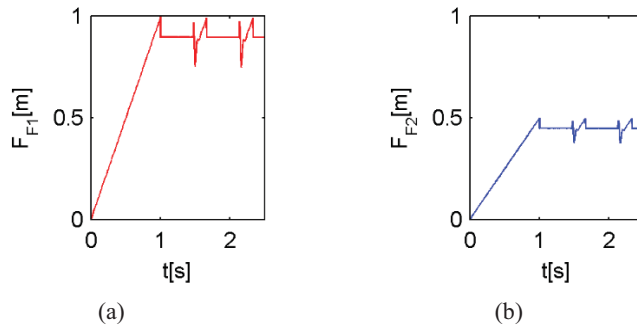


**Figure 3.6.** Simulation results: two-dimensional friction force (a) and bristle trajectory (b)

Next, experiment concerning two-dimensional stick-slip behaviour has been conducted regarding the system shown in Fig. 3.4 (b) for the following fixed parameters:  $x_1(t_0) = x_2(t_0) = 0[\text{m}]$ ,  $v_1(t_0) = v_2(t_0) = 0[\text{m s}^{-1}]$ ,  $z_r(t_0) = 10^{-4}[\text{m}]$ ,  $z_\theta(t_0) = 135[^\circ]$ ,  $v_{y1} = 0,01[\text{m s}^{-1}]$ ,  $v_{y2} = 0,005[\text{m s}^{-1}]$ ,  $v_{y1} = v_{y2} = \text{const}$ . Friction and system parameters are the same as in the previous simulation.



**Figure 3.7.** Time histories of the stick-slip behavior: (a)  $x_1(t)$ , (b)  $x_2(t)$



**Figure 3.8.** Time histories of the friction force: (a)  $F_{F1}(t)$ , (b)  $F_{F2}(t)$

Fig. 3.7 and Fig. 3.8 present simulation results, which capture two-dimensional stick-slip behaviour.

## 5. Conclusions

The new model of two-dimensional friction has been proposed and validated numerically and experimentally. The proposed model is two-dimensional interpretation of the Reset Integrator Model presented by Haessing and Friedland in reference [3] as more computational efficiency version of the bristle model (presented in the same paper). One-dimensional model was converted into two-dimensional space with usage of original two-dimensional bristle concept. The developed by us model allows introducing anisotropy of friction in bristle parameters. It has been tested numerically regarding capturing two-dimensional stick-slip effect and mechanism of smoothing friction by dither. Furthermore the experimental verification has been carried out using special laboratory rig. The carried out laboratory experiments dealt with 2D friction modified and not by dither. Results obtained from the experiment and numerical simulation have shown good agreement, which validates our 2d friction model.

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