On the vibrations of a composite structure with hexagonal structure of a circular inclusions

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Abstract: One of the major advantages of homogenization is a possibility of the generalization of the obtained results. Namely, if a solution to the local problem is found, then without principal problems one may solve not only the analyzed problem, by also a series of related static and dynamic problems, including: linear, quasi-linear, the eigenvalue problems, etc. The mentioned approach has been applied to the eigenvalue problems regarding the perforated structures and periodically non-homogenous 2D constructions with a square mesh of inclusions. In this work we have used theory of averaging to solve the vibrations problem regarding stiffly clamped rectangular membrane with periodically located circular inclusions creating a hexagonal mesh. The relations governing eigenvalues (frequencies) and eigenfunctions have been derived. The derivation of analytical formulas governing membrane eigenforms and frequencies consists of three parts. In the first part the local problem regarding a cell (inclusion) of the composite is studied. Second part is focused on finding main terms of the averaged problem. The third part is aimed at an estimation of the first improvement term with respect to the membrane fundamental frequency.

1. Introduction

The problem of transition of waves in periodic structures consists of a wide spectrum of various questions having roots in different fields of physics and mathematics including mechanics of deformable solids, theory of electromagnetic vibrations and waves, electrotechnics, theory of vibrations, mechanics of composites, theory of crystals, etc.

The first background of the problem investigation has been introduced in 1686 by Newton [31], who studied 1D chain of oscillators while computing the second velocity. Beginning from 1727, a study of 1D chains of oscillators has been carried out by J. Bernoulli and nest by his son D. Bernoulli, who in 1753 formulated the superposition principle i.e. any motion of the vibrating system can be presented as a superposition of its eigenvibrations.

Further development of 1 D models of the wave transition in periodic structures is associated with the works of Cauchy, Rayleigh [34] and Kelvin [25]. In the years 1835-1836 Hamilton obtained remarkable results regarding the light transition in crystals [24]. A review of the state of the art of the development of theory of wave distributions in 1 D chains of oscillators can be found in reference [16].

Among the fundamental works published in the middle of the previous century we mention monographs of Brillouin [14, 15] and Brillouin and Parodi, where not only the mathematical theory of 1 D chains is presented, but also systems with the cell of periodicity having a few degrees-of-freedom (DOFs) have been considered. In reference [16] the obtained results of investigations are generalized into 3D case and a solution to 3D wave equations is given. Chains of coupled particles have been considered by Ashcroft and Solid [8]. Born and Karman [12] have applied this concept of chains to model wave transitions in crystals from a point of view of solid mechanics. Idealization idea introduced through coupled chains of particles/masses has been also applied in other scientific disciplines like atomic and molecular dynamics in physics, in chemistry and biology as well as in mechanics for modeling rod and beam constructions with added particles, in quasi-periodic systems, etc. [see for, instance, 17, 19, 28, 29, 35].

In the series of publications [23, 30, 36, 37] the transition and localization of elastic waves in periodic composite materials have been studied.

On the other hand it is well known that the theory of averaging belongs to one of the effective mathematical tools for modeling physical processes of different kind in periodic non-homogenous structures [9, 10, 27]. One of the major advantages of the homogenization is a possibility of the generalization of the obtained results. Namely, if a solution to the local problem is found, then without principal problems one may solve not only the analyzed problem, but also a series of related static and dynamic problems, including: linear, quasi-linear, the eigenvalue problems, etc. The mentioned approach has been applied to the eigenvalue problems regarding the perforated structures and periodically non-homogenous 2D constructions with the square mesh of inclusions in references [4-7].

Application of the asymptotic homogenization [2, 3] is based on the procedure of multi-scale series and yields reliable solutions regarding long waves for the low frequencies level [1, 13,21]. The discussed approach has been applied in the linear case in references [1, 18, 20]. On the other hand, in references [11,22,33] the method of asymptotic homogenization has been applied in the case of large deformations with a successive linear distribution of the elastic wave.

In this work, based on the averaging theory and the multi-scale series, the analytical results regarding dynamic features of the membrane vibrations having periodically located circular inclusions in the form of a hexagonal mesh are obtained. In particular, relations yielding eigenfrequencies and the associated eigenfunctions including first improvement terms are derived.

2. Problem formulation in terms of averaging theory

We consider the eigenvalue problem of a vibrating rectangular membrane clamped along its contour treated as a composite structure with periodically located circled inclusions forming the hexagonal mesh (Figure 1).
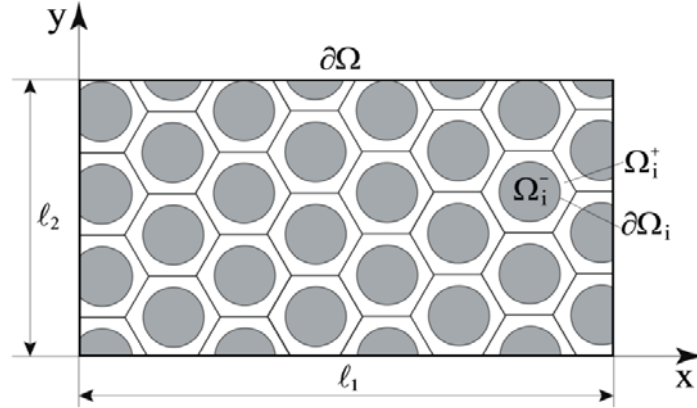

Figure 1. Composite material with hexagonal structure of inclusions

In the general case, the membrane eigenvalue problem can be formulated in the following form:

$$c^{+2}\left(\frac{\partial^2 u^+}{\partial x^2}+\frac{\partial^2 u^+}{\partial y^2}\right)=\frac{\partial^2 u^+}{\partial t^2} \quad \text{in } \Omega_i^+; \tag{1}$$

$$c^{-2}\left(\frac{\partial^2 u^-}{\partial x^2}+\frac{\partial^2 u^-}{\partial y^2}\right)=\frac{\partial^2 u^-}{\partial t^2} \quad \text{in } \Omega_i^-; \tag{2}$$

$$u^+=u^-, \quad c^{+2}\frac{\partial u^+}{\partial n}=c^{-2}\frac{\partial u^-}{\partial n} \quad \text{on } \partial\Omega_i; \tag{3}$$

$$u^{\pm}=0 \quad \text{on } \partial\Omega; \tag{4}$$

$$u^{\pm}=f^{\pm}(x,y), \quad \frac{\partial u^{\pm}}{\partial t}=F^{\pm}(x,y) \quad \text{for } t=0, \tag{5}$$

where: u –transversal displacement of the membrane points; $c^2=\dfrac{p}{\rho}$, p –stress inside the membrane, ρ –surface density; n –external normal to the inclusion contour.

Solution to the problem (1)-(5) is assumed to be as follows:

$$u^{\pm}(x, y, t) = u^{\pm}(x, y)e^{i\varpi t}, \tag{6}$$

where ϖ –stands for a circular frequency.

Then, owing to (6) problem (1)-(3) can be transformed to the following form

$$\frac{\partial^2 u^+}{\partial x^2} + \frac{\partial^2 u^+}{\partial y^2} + \varpi u^+ = 0 \text{ in } \Omega_i^+; \tag{7}$$

$$\lambda\left(\frac{\partial^2 u^-}{\partial x^2} + \frac{\partial^2 u^-}{\partial y^2}\right) + \varpi u^- = 0 \text{ in } \Omega_i^-; \tag{8}$$

$$u^+ = u^-, \quad \frac{\partial u^+}{\partial n} = \lambda \frac{\partial u^-}{\partial n} \text{ on } \partial\Omega_i, \tag{9}$$

where: $\lambda = \dfrac{c^{-2}}{c^{+2}}$; $\varpi = \dfrac{\omega^2}{c^{+2}}$.

Following the theory of averaging [9, 10] and applying two-scale techniques [31] the solutions to the problem (7)-(9), (4) in the form of asymptotic series regarding a small parameter characterizing the structural period are assumed in the following form

$$u^{\pm} = u_0(x, y) + \varepsilon\left[u_{10}(x, y) + u_1^{\pm}(x, y, \xi, \eta)\right] + \\ + \varepsilon^2\left[u_{20}(x, y) + u_2^{\pm}(x, y, \xi, \eta)\right] + \ldots \tag{10}$$

where: ξ, η stand for the fast variables, and $\xi = \dfrac{x}{\varepsilon}$, $\eta = \dfrac{y}{\varepsilon}$.

The frequency is also presented in the form of the asymptotic series:

$$\varpi = \varpi_0 + \varepsilon \varpi_1 + \varepsilon^2 \varpi_2 + \ldots. \tag{11}$$

After splitting procedure regarding ε being applied to relations (7), (8), the following infinite recursive set of equations is obtained

$$\varepsilon^{-1}: \quad \begin{aligned} &\frac{\partial^2 u_1^+}{\partial \xi^2} + \frac{\partial^2 u_1^+}{\partial \eta^2} = 0; \\ &\frac{\partial^2 u_1^-}{\partial \xi^2} + \frac{\partial^2 u_1^-}{\partial \eta^2} = 0; \end{aligned} \tag{12}$$

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$$\varepsilon^0 : \quad \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + 2\frac{\partial^2 u_1^+}{\partial x \partial \xi} + 2\frac{\partial^2 u_1^+}{\partial y \partial \eta} + \frac{\partial^2 u_2^+}{\partial \xi^2} + \frac{\partial^2 u_2^+}{\partial \eta^2} + \varpi_0 u_0 = 0;$$

$$\lambda\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + 2\frac{\partial^2 u_1^-}{\partial x \partial \xi} + 2\frac{\partial^2 u_1^-}{\partial y \partial \eta} + \frac{\partial^2 u_2^-}{\partial \xi^2} + \frac{\partial^2 u_2^-}{\partial \eta^2} \right) + \varpi_0 u_0 = 0;$$

(13)

$$\varepsilon : \quad \frac{\partial^2 u_1^+}{\partial x^2} + \frac{\partial^2 u_1^+}{\partial y^2} + \frac{\partial^2 u_{10}}{\partial x^2} + \frac{\partial^2 u_{10}}{\partial y^2} + 2\frac{\partial^2 u_2^+}{\partial x \partial \xi} + 2\frac{\partial^2 u_2^+}{\partial y \partial \eta} + \frac{\partial^2 u_3^+}{\partial \xi^2} + \frac{\partial^2 u_3^+}{\partial \eta^2} +$$

$$+ \varpi_1 u_0 + \varpi_0 \left(u_1^+ + u_{10} \right) = 0;$$

$$\lambda\left(\frac{\partial^2 u_1^-}{\partial x^2} + \frac{\partial^2 u_1^-}{\partial y^2} + \frac{\partial^2 u_{10}}{\partial x^2} + \frac{\partial^2 u_{10}}{\partial y^2} + 2\frac{\partial^2 u_2^-}{\partial x \partial \xi} + 2\frac{\partial^2 u_2^-}{\partial y \partial \eta} + \frac{\partial^2 u_3^-}{\partial \xi^2} + \frac{\partial^2 u_3^-}{\partial \eta^2} \right) +$$

$$+ \varpi_1 u_0 + \varpi_0 \left(u_1^- + u_{10} \right) = 0;$$

(14)

The compatibility relations (9) takes the form:

$$\varepsilon^1 : \quad u_1^+ = u_1^- ;$$

(15)

$$\varepsilon^2 : \quad u_2^+ = u_2^- ;$$

(16)

$$\varepsilon^0 : \quad \frac{\partial u_1^+}{\partial \bar{n}} + \frac{\partial u_0}{\partial n} = \lambda\left(\frac{\partial u_1^-}{\partial \bar{n}} + \frac{\partial u_0}{\partial n} \right);$$

(17)

$$\varepsilon^1 : \quad \frac{\partial u_2^+}{\partial \bar{n}} + \frac{\partial u_1^+}{\partial n} + \frac{\partial u_{10}}{\partial n} = \lambda\left(\frac{\partial u_2^-}{\partial \bar{n}} + \frac{\partial u_1^-}{\partial n} + \frac{\partial u_{10}}{\partial n} \right),$$

(18)

where $\dfrac{\partial}{\partial \bar{n}}$, $\dfrac{\partial}{\partial n}$ are derivatives along the external normal to an inclusion contour regarding fast and slow variables, respectively:

$$\frac{\partial}{\partial \bar{n}} = \frac{\partial}{\partial \xi}\cos\alpha + \frac{\partial}{\partial \eta}\cos\beta ,$$

(19)

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\cos\beta .$$

(20)

Therefore, the problem of finding a solution to the complex space governed by equations (7)–(9), (4) is split to a series of problems in the spaces of essentially simpler geometry in comparison to the input problem.

3. Derivation of analytical formulas governing membrane eigenforms and frequencies

Owing to the general procedure of the homogenization method a solution to the problem (7)–(9), (4) can be divided into three parts:
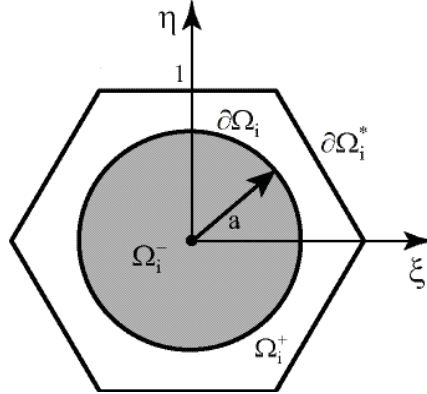

Figure 2. Characteristic structure of the composite cell: Ω_i^+ –matrix area; Ω_i^- –inclusion area.

(i) In the first part a solution to the local problem is defined [9, 10], i.e. we study a periodically repeated cell of the composite (see Figure 2):

$$\frac{\partial^2 u_1^{\pm}}{\partial \xi^2} + \frac{\partial^2 u_1^{\pm}}{\partial \eta^2} = 0 \text{ in } \Omega_i^{\pm}, \tag{21}$$

$$u_1^+ = u_1^-, \quad \frac{\partial u_1^+}{\partial \overline{n}} - \lambda \frac{\partial u_1^-}{\partial \overline{n}} = (\lambda - 1)\frac{\partial u_0}{\partial n} \text{ on } \partial \Omega_i, \tag{22}$$

$$u_1^+ = 0 \text{ on } \partial \Omega_i^*. \tag{23}$$

(ii) The second part is aimed at finding main parts of the eigenfunctions and frequencies of the averaged problems. The averaged equation is obtained through application of the following averaging operator

$$\tilde{\Phi}(x, y) = \frac{1}{|\Omega_i^*|}\left[\iint\limits_{\Omega_i^+} \Phi^+(x, y, \xi, \eta)\, d\xi \partial \eta + \lambda \iint\limits_{\Omega_i^-} \Phi^-(x, y, \xi, \eta)\, d\xi \partial \eta \right] \tag{24}$$

to relations (2.13), and the input problem takes the following form:

$$\bar{q}\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}\right)+$$

$$\frac{1}{|\Omega_i^*|}\left[\iint_{\Omega_i^+}\left(\frac{\partial^2 u_1^+}{\partial x\partial\xi}+\frac{\partial^2 u_1^+}{\partial y\partial\eta}\right)d\xi\partial\eta+\lambda\iint_{\Omega_i^-}\left(\frac{\partial^2 u_1^-}{\partial x\partial\xi}+\frac{\partial^2 u_1^-}{\partial y\partial\eta}\right)d\xi\partial\eta\right]+\varpi_0 u_0=0 \tag{25}$$

where $\Omega_i^*=\Omega_i^+\bigcup\Omega_i^-$; $\bar{q}=\dfrac{|\Omega_i^+|+\lambda|\Omega_i^-|}{|\Omega_i^*|}$ –averaged parameter in the sense of Foight.

Taking into account the relations for u_1^+, u_1^- which define the problem on the cell (21)–(23), the averaged problem can be transformed to the following one:

$$q_x\frac{\partial^2 u_0}{\partial x^2}+q_y\frac{\partial^2 u_0}{\partial y^2}+\bar{q}\,\varpi_0 u_0=0 \text{ in } \Omega^*, \tag{26}$$

$$u_0=0 \text{ on } \partial\Omega, \tag{27}$$

where: $\Omega^*:\begin{Bmatrix}0\le x\le\ell_1\\0\le y\le\ell_2\end{Bmatrix}$ –smooth membrane space with the reduced characteristics; q_x, q_y– averaged parameters of the following form:

$$q_x=\bar{q}+\frac{1}{|\Omega_i^*|}\left(\iint_{\Omega_i^+}\frac{\partial u_{1(1)}^+}{\partial\xi}d\xi\partial\eta+\lambda\iint_{\Omega_i^-}\frac{\partial u_{1(1)}^-}{\partial\xi}d\xi\partial\eta\right); \tag{28}$$

$$q_y=\bar{q}+\frac{1}{|\Omega_i^*|}\left(\iint_{\Omega_i^+}\frac{\partial u_{1(2)}^+}{\partial\eta}d\xi\partial\eta+\lambda\iint_{\Omega_i^-}\frac{\partial u_{1(2)}^-}{\partial\eta}d\xi\partial\eta\right), \tag{29}$$

where: $u_{1(i)}^\pm$ $(i=1,2)$ –are the solutions to the local problems (21)–(23), estimated with accuracy up to constant multipliers corresponding to the slow solution component:

$$u_1^\pm=u_{1(1)}^\pm(\xi,\eta)\frac{\partial u_0}{\partial x}+u_{1(2)}^\pm(\xi,\eta)\frac{\partial u_0}{\partial y}.$$

Solution to the eigenvalue problem (26), (27) in the smooth space of the rectangular membrane with reduced physical characteristics can be formulated in the following form

$$u_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} S_{mn} \sin\frac{m\pi x}{\ell_1} \sin\frac{n\pi y}{\ell_2}, \tag{30}$$

$$\varpi_0 = \frac{\pi^2}{\bar{q}}\left(q_x\left(\frac{m}{\ell_1}\right)^2 + q_y\left(\frac{n}{\ell_2}\right)^2\right), \tag{31}$$

where constants S_{mn} $(m, n = 1, 2, ...)$ are defined by the initial conditions (5).

(iii) The third part of our approach is focused on estimation of the first improvement term regarding the frequency ϖ_1. This requires finding one more approximation to the function $u^{\pm}$, i.e. we need to find functions $u_2^{\pm}$ as a solution to the following compatibility problem:

$$\frac{\partial^2 u_2^+}{\partial \xi^2} + \frac{\partial^2 u_2^+}{\partial \eta^2} = -\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + 2\frac{\partial^2 u_1^+}{\partial x \partial \xi} + 2\frac{\partial^2 u_1^+}{\partial y \partial \eta}\right) - \varpi_0 u_0 \text{ in } \Omega_i^+; \tag{32}$$

$$\lambda\left(\frac{\partial^2 u_2^-}{\partial \xi^2} + \frac{\partial^2 u_2^-}{\partial \eta^2}\right) = -\lambda\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + 2\frac{\partial^2 u_1^-}{\partial x \partial \xi} + 2\frac{\partial^2 u_1^-}{\partial y \partial \eta}\right) - \varpi_0 u_0 \text{ in } \Omega_i^-; \tag{33}$$

$$u_2^+ = u_2^-, \quad \frac{\partial u_2^+}{\partial \bar{n}} - \lambda\frac{\partial u_2^-}{\partial \bar{n}} = (\lambda - 1)\frac{\partial u_{10}}{\partial n} + \lambda\frac{\partial u_1^-}{\partial n} - \frac{\partial u_1^+}{\partial n} \text{ on } \partial\Omega_i; \tag{34}$$

$$u_2^+ = 0 \text{ on } \partial\Omega_i^*. \tag{35}$$

Observe that the structure of the problems regarding the first and second approximations of the cell is identical, and they differ only in the amount an even component, which does not play the principal role, i.e. it does not introduce an essential input to the averaging, and hence it can be omitted while estimating the frequency ϖ_1 [6,7].

Consequently, relations for the functions $u_2^{\pm}$, yielded by solving the problems (32)–(35), can be presented in the following general way:

$$u_2^{\pm} = u_1^{\pm}(u_0 \to u_{10}) + U_2^{\pm}(\xi, \eta),$$

where $U_2^{\pm}(-\xi, -\eta) = U_2^{\pm}(\xi, \eta)$.

Owing to the latter remark, the averaged problem of the second approximation is governed by the averaged equation obtained by application of the averaging operator (24) to relations (34):

$$q_x \frac{\partial^2 u_{10}}{\partial x^2} + q_y \frac{\partial^2 u_{10}}{\partial y^2} + \bar{q}\left(\varpi_0\, u_{10} + \varpi_1\, u_0\right) = 0 \ \text{ in } \Omega^*, \tag{36}$$

with the following boundary condition

$$u_{10} = -\tilde{u}_1 \ \text{ on } \partial\Omega, \tag{37}$$

where $\tilde{u}_1$ is the averaged part of the function $u_1^{\pm}\left(x, y, \xi, \eta\right)$.

It should be emphasized that the relation (36) includes two unknown functions: u_{10} – slow solution of the ε^1 order and the first improvement term to the frequency ϖ_1. In order to find the frequency ϖ_1 we need to transform (36) applying the known scheme (see [26]). Namely, we multiply (30) by u_0 and next we integrate this equation by parts regarding the space Ω^*, which taking into account (26) and boundary condition (27) yields the following relation:

$$\varpi_1 \bar{q} \int_0^{\ell_1}\int_0^{\ell_2} u_0^2\, dxdy - q_x \int_0^{\ell_2} \frac{\partial u_0}{\partial x} u_{10} \bigg|_{x=0}^{x=\ell_1} dy - q_y \int_0^{\ell_1} \frac{\partial u_0}{\partial y} u_{10} \bigg|_{y=0}^{y=\ell_2} dx = 0. \tag{38}$$

Therefore, if

$$u_{10} = -\tilde{u}_1 = 0 \ \text{ on } \partial\Omega,$$

then $\varpi_1 = 0$, and the series responsible for eigenfrequency begins with ϖ_2, i.e. the term of order ε^2.

In the case when u_{10} does not satisfy the boundary conditions on the external membrane contour, i.e. $u_{10} \neq 0$ on $\partial\Omega$, we get non-zero first improvement term regarding the eigenfrequency, which taking into account (38) has the following form:

$$\varpi_1 = \frac{q_x \int_0^{\ell_2} \phi(y)\, dy + q_y \int_0^{\ell_1} \varphi(x)\, dx}{\bar{q} \int_0^{\ell_1}\int_0^{\ell_2} u_0^2\, dxdy}, \tag{39}$$

where

$$\phi(y) = \frac{\partial u_0}{\partial x} u_{10} \bigg|_{x=0}^{x=\ell_1}; \quad \varphi(x) = \frac{\partial u_0}{\partial y} u_{10} \bigg|_{y=0}^{y=\ell_2}.$$ (40)

Subsequent term of the series (10), (11) can be found by analogous scheme and its estimation does not require principal difficulties.

4. Conclusions

The theory of averaging has been applied in order to solve the problem of vibrations of the rectangular membrane stiffly clamped along its contour, representing the composite structure with periodically located circular inclusions within the hexagonal mesh.

In the general case, the fundamental analytical relations of smooth components of the eigenfunctions and the eigenfrequencies (30), (31) as well as of their first fast oscillating terms (21)-(23), (39), (40) have been derived.

Practical realization of the proposed approach depends strongly on finding a solution of the problem regarding the cell, i.e. in a correct estimation of the averaged characteristics of the non-homogenous structure.

Depending on the values of the physical and geometric characteristics of the composite a solution to the local problem can be obtained either by an asymptotic series or by combined analytical-numerical approaches.

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