Vibration of functionally graded shallow shells with complex shape

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Abstract: The method for studying the geometrically nonlinear vibrations of functionally graded shallow shells with a complex planform is proposed. Composite shallow shells made from a mixture of ceramic and metal are considered. In order to take into account varying of the volume fraction of ceramic the power law is accepted. Formulation of the problem is carried out using the refined geometrically nonlinear theory of shallow shells of the first order (Timoshenko's type). The R-functions theory, variational Ritz's method, procedure by Bubnov-Galerkin and Runge-Kytta method are used in the developed approach. A distinctive feature of the proposed approach is the method of reducing the initial nonlinear system of equations of motion for partial derivatives to a nonlinear system of ordinary differential equations. According to the developed approach first it is necessary to solve linear vibration problem. Further to solve elasticity problems for inhomogeneous differential equations with right hand side, containing eigen functions. Obtained solutions of these problems are applied for representation of unknown functions of the nonlinear problem. Application of the theory of R-functions on every step allows us to extend the proposed approach to the shell with arbitrary shape of plan and different kinds of boundary condition. The proposed method is validated by investigation of test problems for shallow shells with rectangular and elliptical planform and applied to new vibration problems for shallow shells with complex planform.

1.Introduction

Structure elements simulated by shallow shells are widely used in various engineering fields: mechanical, aerospace, marine, military, civil engineering and others. Such elements can have a various planform, boundary conditions including mixed ones and types of curvature. In order to improve the strength of the modern design the new class of the composite materials, functionally graded materials (FGM) are applied. In spite of FGM are inhomogeneous materials they have essential advantage over composite ones, because they possess smooth and continuous variation of material properties in the thickness direction. It eliminates the stress concentration presented in laminated structure. But analysis of functionally graded (FG) shells is more complicated than homogeneous material structures. It is connected with the following fact. Governing equations of the shallow shells made of FGM are partial differential equations with variable coefficients. As known solving of such equations supplemented by various boundary conditions in case of an arbitrary planform is very difficult problem. In addition the problem is compounded if FG shallow shells performs vibration with large amplitudes. This class of problem is challenging and there exists numerous of investigations devoted to analysis of dynamical behavior of the FG plates and shells. This is especially true for linear problems [1-5]. In the last decade in addition to the linear vibration analysis nonlinear free and force vibrations of the FG shells have been extensively studied [6-8]. A complete survey on the linear and nonlinear vibrations of FG plates and shells can be found in the following papers [3-7]. Note that in the aforementioned papers nonlinear vibration of simply supported or clamped FG structures with rectangular, skew or circle planform was analyzed by different numerical methods such as finite element method (FEM), Differential Quadrate method, domain decomposition approach, Haar Wavelet Discretization method, modified Fourier-Ritz approach and others. Some survey on vibration concerning the open revolution shells can be found in [5, 6, 8]. The analysis of published literature on the nonlinear vibration of FG shallow shells is restricted to simple plan-form and classical boundary conditions. But in practice FG shells with an arbitrary planform and different boundary conditions are widely used. Consequently it is important to develop universal and effective methods for investigation of nonlinear vibration of functionally graded shallow shells with complex planform and different boundary conditions. Earlier in papers [9-12] the original meshless method based on application of the R-functions theory, variational Ritz's method, procedure by Bubnov-Galerkin, method by Runge-Kutta has been proposed.

The main aim of this paper is development of this efficient and enough universal method to the new class of the nonlinear problems-geometrically nonlinear vibrations of the FG shallow shells with an arbitrary planform. The proposed method is validated by investigation of the test problems for shallow shells with rectangular and elliptical plan-form and applied to the new vibration problems for shallow shells with complex planform.

2. Mathematical formulation

Consider functionally graded shallow shell with uniform thickness *h* made of a mixture of ceramics and metals. It is assumed that shells can have an arbitrary planform as shown in Fig 1. As known [1-3, 6-8] in FGM structures material properties are proportional to the volume fraction of the constituent materials

$$E(z) = (E_c - E_m)V_c + E_m, \quad \nu(z) = (\nu_c - \nu_m)V_c + \nu E_m, \quad \rho(z) = (\rho_c - \rho_m)V_c + \rho_m\,, \qquad (1)$$

where E, ν and ρ are Young's modulus, Poisson's ratio and mass density respectively. The subscripts *c* and *m* denote the ceramic and metallic constituents. Ceramic volume fraction is denoted by V_c. In this work value V_c is expressed as

$$V_c = \left(\frac{z}{h} + \frac{1}{2}\right)^k \quad . \qquad (2)$$

Let us note that the volumes of all constituents materials should add to up one, i.e. $V_c + V_m = 1$. Index k $(0 \le k \prec \infty)$ denotes the volume fraction exponent, z is the distance between a current point and mid-surface. In particular case when the power index k is equal to zero then we obtain homogeneous material –ceramic, but if k approaches infinity then we have pure metallic shell.

According to the nonlinear first order shear deformation theory of shallow shell (FSDT), the displacements components u_1, u_2, u_3 at a point (x, y, z) are expressed as functions of the middle surface displacements u, v and w in the Ox, Oy and Oz directions and the independent rotations ψ_x, ψ_y of the transverse normal to the middle surface about the Oy and Ox axes respectively as [2,6]:

$$u_1 = u + z\psi_x, \quad u_2 = v + z\psi_y, \quad u_3 = w\ .$$

Strain components $\varepsilon = \{\varepsilon_{11}; \varepsilon_{22}; \varepsilon_{12}\}^T$, $\chi = \{\chi_{11};\ \chi_{22};\ \chi_{12}\}^T$ at an arbitrary point of the shallow shell are:

$$\varepsilon_{ij} = \varepsilon_{ij}^{L} + \varepsilon_{ij}^{ND}, \quad (i, j = 1{,}2),$$

where

$$\varepsilon_{11}^{L} = u,_x + w/R_x \quad \varepsilon_{22}^{L} = v,_y + w/R_y \quad \varepsilon_{12}^{L} = u,_y + v,_x\ ,$$

$$\varepsilon_{11}^{ND} = \frac{1}{2} w,_x^2\ , \quad \varepsilon_{22}^{ND} = \frac{1}{2} w,_y^2\ , \quad \varepsilon_{12}^{ND} = w,_x w,_y\ ,$$

$$\varepsilon_{13} = w,_x + \psi_x, \quad \varepsilon_{23} = w,_y + \psi_y\ , \chi_{11} = \psi_x,_x\ , \quad \chi_{22} = \psi_y,_y \quad \chi_{12} = \psi_x,_y + \psi_y,_x\ . \tag{3}$$

The in-plane force resultant vector $N = (N_{11}, N_{22}, N_{12})^T$, bending and twisting moments resultant vector $M = (M_{11}, M_{22}, M_{12})^T$ and transverse shear force resultant $Q = (Q_x, Q_y)^T$ are calculated by integration along the Oz -axes and in the matrix form are defined as

$$N = [A]\varepsilon + [B]\chi, \quad M = [B]\varepsilon + [D]\chi\ , \tag{4}$$

where

$$([A],[B],[D]) \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E(z)}{1-\nu^2(z)} [C](1, z, z^2)dz\ , \quad [C] = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \tag{5}$$

Transverse shear force resultants $Q_x,\quad Q_y$ are defined as:

$$Q_x = K_s^2 A_{33}\varepsilon_{13}, \quad Q_y = K_s^2 A_{33}\varepsilon_{23} , \tag{6}$$

where K_s^2 denotes the shear correction factor. In this paper it will be selected by 5/6.

Mass density ρ is also defined by integration along thickness:

$$I_0 = \int_{-h/2}^{h/2} \rho(z)dz = \left(\rho_m + \frac{\rho_c - \rho_m}{k+1}\right)h .$$

The strain and kinetic energy of the FG shells is given by

$$U = \frac{1}{2}\iint_{\Omega} \left(N_{11}\varepsilon_{11} + N_{22}\varepsilon_{22} + N_{12}\varepsilon_{12} + M_{11}\chi_{11} + M_{22}\chi_{22} + M_{12}\chi_{12} Q_x\varepsilon_{13} + Q_y\varepsilon_{23}\right)d\Omega ,$$

$$T = \frac{1}{2}\iint_{\Omega} I_0\left(u,_t^2 + v,_t^2 + w,_t^2\right) + 2I_1\left(u,_t \psi_x,_t + v,_t \psi_y,_t\right) + I_2\left(\psi_x,_t^2 + \psi_y,_t^2\right)dxdy , \tag{7}$$

where

$$I_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z)\, zdz = \frac{(\rho_c - \rho_m)k}{2(k+1)(k+2)} h^2 ,$$

$$I_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z)\, z^2 dz = \left(\frac{\rho_m}{12} + (\rho_c - \rho_m)\left(\frac{1}{k+3} - \frac{1}{k+2} + \frac{1}{4(k+4)}\right)\right)h^3 \tag{8}$$

Applying the method by Ostrogradsky-Hamilton the equations of motion can be obtained. These equations are supplemented by the corresponding boundary conditions.

3. Solution method

To solve this problem we use the approach proposed in [9]. To implement this approach it is first necessary to solve the linear problem of free vibrations of FG shallow shell. For this purpose the vector of unknown functions is represented as

$$\vec{U}(\bar{u}(x,y,t), \bar{v}(x,y,t), \bar{w}(x,y,t), \overline{\psi_x}(x,y,t), \overline{\psi_y}(x,y,t)) = \vec{U}(u(x,y), v(x,y), w(x,y), \psi_x(x,y), \psi_y(x,y)) \sin \lambda t , \tag{9}$$

where λ is a vibration frequency. Applying the principle of Ostrogradsky-Hamilton we get the variational equation in the form

$$\partial(U_{\max} - \lambda T_{\max}) = 0 , \tag{10}$$

where expressions for strain U and kinetic energy T are defined by relations:

$$U_{\max} = \frac{1}{2}\iint_{\Omega} \left(N_{11}^L\varepsilon_{11}^L + N_{22}^L\varepsilon_{22}^L + N_{12}^L\varepsilon_{12}^L + M_{11}^L\chi_{11} + M_{22}^L\chi_{22} + M_{12}^L\chi_{12} + Q_x\varepsilon_{13} + Q_y\varepsilon_{23}\right)d\Omega ,$$

$$T_{\max} = \frac{1}{2}\iint_{\Omega} I_0\left(u^2 + v^2 + w^2\right) + 2I_1\left(u\psi_x + v\psi_y\right) + I_2\left(\psi_x{}^2 + \psi_y{}^2\right) dxdy \,.$$

Minimization of the functional (10) will be performed using the Ritz's method and we will build the necessary sequence of coordinate functions by the R-functions theory [13, 14].

To solve the nonlinear problem according to the approach proposed in [9] we represent unknown functions in the form of an expansion in the eigen functions $w_i^{(c)}(x,y)$, $u_i^{(c)}(x,y)$, $v_i^{(c)}(x,y)$, $\psi_{xi}^{(c)}(x,y)$, $\psi_{yi}^{(c)}(x,y)$ of the linear problem with coefficients $y_k(t)$ depending on the time:

$$w = \sum_{i=1}^{n} y_i(t) w_i^{(c)}(x,y), \; \psi_x = \sum_{i=1}^{n} y_i(t) \psi_{xi}^{(c)}(x,y), \; \psi_y = \sum_{i=1}^{n} y_i(t) \psi_{yi}^{(c)}(x,y),$$

$$u = \sum_{i=1}^{n} y_i(t) u_i^{(c)}(x,y) + \sum_{i=1}^{n}\sum_{j=1}^{n} y_i y_j u_{ij} \;, \; v = \sum_{i=1}^{n} y_i(t) v_i^{(c)}(x,y) + \sum_{i=1}^{n}\sum_{j=1}^{n} y_i y_j v_{ij}, \tag{11}$$

Note that the functions u_{ij}, v_{ij} should be found from the following set of differential equations:

$$\begin{cases} L_{11}\left(u_{ij}\right) + L_{12}\left(v_{ij}\right) = -Nl_1^{(2)}\left(w_i^{(c)}, w_j^{(c)}\right), \\ L_{21}\left(u_{ij}\right) + L_{22}\left(v_{ij}\right) = -Nl_2^{(2)}\left(w_i^{(c)}, w_j^{c}\right). \end{cases} \tag{12}$$

where

$$Nl_1^{(2)}\left(w_i^{(c)}, w_j^{(c)}\right) = w_i^{(c)},_x L_{11} w_j^{(c)} + w_i^{(c)},_y L_{12} w_j^{(c)},$$

$$Nl_2^{(2)}\left(w_i^{(c)}, w_j^{(c)}\right) = w_i^{(c)},_x L_{12} w_j^{(c)} + w_i^{(c)},_y L_{22} w_j^{(c)}.$$

Operators $L_{11}, L_{22}, L_{12}, L_{21}$ are presented bellow

$$L_{11} = C_{11}(\;),_{xx} + 2C_{16}(\;),_{xy} + C_{66}(\;),_{yy} \,,\; L_{22} = C_{66}(\;),_{xx} + 2C_{26}(\;),_{xy} + C_{22}(\;),_{yy} \,,$$
$$L_{12} = L_{21} = C_{16}(\;),_{xx} + (C_{12} + C_{66})(\;),_{xy} + C_{26}(\;),_{yy} \,.$$

The system of equations (12) can be solved via RFM virtually for any planform and various kinds of boundary conditions. Substituting expression (11) for the functions u, v, w, ψ_x, ψ_y in the equations of motion and applying the Bubnov-Galerkin procedure, we obtain the following system of nonlinear ordinary differential equations for the unknown functions $y_j(t)$:

$$y_r''(t)+\omega_{Lr}^2 y_r(t)+\sum_{i,j=1}^{n}\beta_{ij}^{(r)}y_i(t)y_j(t)+\sum_{i,j,k=1}^{n}\gamma_{ijk}^{(r)}y_i(t)y_j(t)y_k(t)=0\ . \qquad (13)$$

The expressions for the coefficients $\beta_{ij}^{(r)},\gamma_{ijk}^{(r)}$ are the same with [23, 25, 28]

The solution of equation (13) can be found by various approximation methods. In this paper we used the Runge-Kutta method. At the same time in the numerical implementation we were limited to only one mode. Thus, instead of the system of equations (13) we find the solution of one differential equation.

4. Numerical results

In order to verify the proposed approach we have solved some test problems and compared obtained results with available ones. In particular case we considered FG shallow simply-supported shells with square planform. Obtained results we compare with results presented in [1, 7, 8] for mixture Al/Al_2O_3 and various shallowness ratios and different thicknesses. Comparative analysis has shown that the results obtained using the refined theory of the first order (RFM, FSDT) are almost the same as presented in [8]. A deviation from the results obtained by theory of the higher order (HSDT) [1] does not exceed 4%. Deviations results obtained by using the classical theory (RFM, CST) with the results of [7] do not exceed 2%. In general, it should be noted that the classical theory leads in most cases to overestimate the fundamental frequencies compared with the refined theory.

The second testing example considers a clamped FG moderately thick doubly-curved shallow shell of elliptical planform with $R_x/R_y=1,\ a/b=2,\ h/2a=0.1,\ 2a/R_x=0.2$, where a and b are ellipse semi axes. Two types of FG mixture are analyzed. Properties of the FG mixture are the same with [8]:

FG1: $Al/Al_2O_3 : E_m/E_c=70/380GPa,\ \nu_m=\nu_c=0.3,\ \rho_m/\rho_c=2707/3800kg/m^3$ **;** (14)

FG2: $Al/ZrO_2 : E_m/E_c=70/151GPa,\ \nu_m=\nu_c=0.3,\quad \rho_m/\rho_c=2707/3000kg/m^3$. (15)

Fig. 1 shows comparison of the fundamental frequency parameters $\Omega_L=\lambda_1 a^2 h\sqrt{\rho_c/E_c}$ as a function of the volume fraction exponent *k*. It can be observed that presented results are in excellent agreement with those reported by S.M.Chorfi and A.Houmat [8].

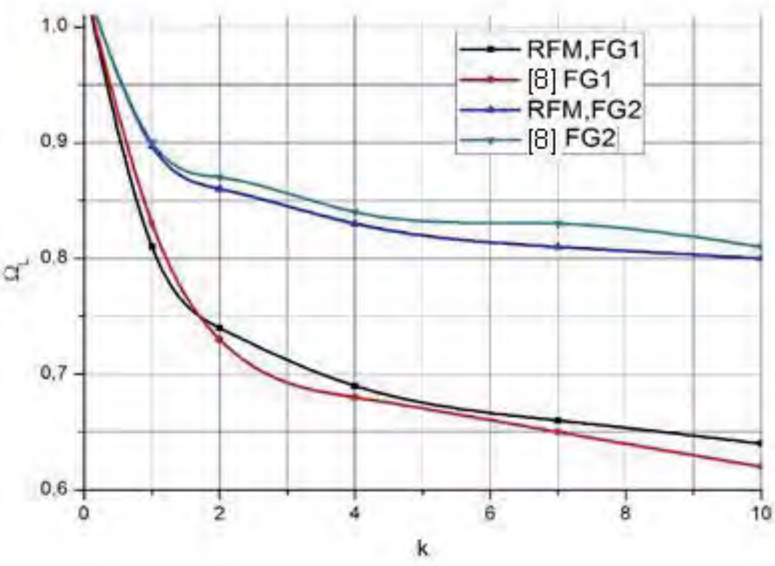


Fig. 1. Influence of the power-law exponent *k* on the frequency parameter
$\Omega_L = \lambda_1 a^2 h \sqrt{\rho_c / E_c}$ of the FG spherical shallow shell of elliptical planform

The non-linear free vibrations of the clamped spherical shell with elliptical planform will be investigated also. Fig 2 shows the non-linear frequency amplitude relationships (backbone curves).

In Fig 2. the backbone curves are compared with available [8] for mixture FG1 and two values of the parameters $k = 0, k = 1$, and for mixture FG2 and values of the parameters $k = 10, k = 100$. Comparison of the backbone curves with the results of [8] confirms the accuracy of the proposed approach. The results are practically the same within the accuracy of the plot. Deviation does not exceed 1%.

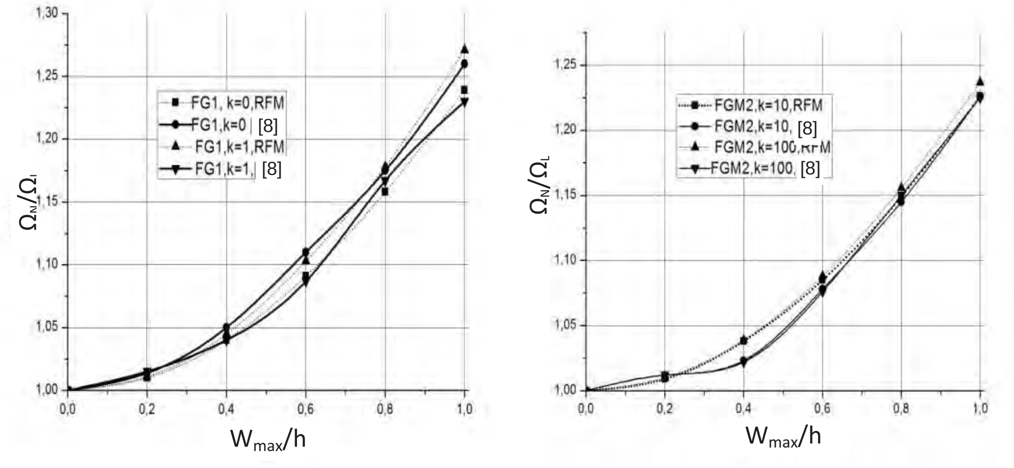


Fig.2 Comparison between frequency response curves of the clamped spherical shell with elliptical planform with results presented by S.M.Chorfi and A.Houmat [8]

To contribute to new results and to illustrate the versatility and efficiency of the proposed method let us complicate the elliptical planform of shells. Assume that shell has the planform presented in Fig. 3. Geometrical parameters are

$R_x / R_y = 1, b/2a = 0.5, h/2a = 0.1, 2a/R_x = 0.2, b_1/2a = 0.35, a_1/2a = 0.2$.

Material properties of the mixture (FG1, FG2) are the same as in task 2. Suppose that shell is also clamped. Then the solution structure [13] may be taken in the following form

$$u = \omega\Phi_1, \quad v = \omega\Phi_2, \quad w = \omega\Phi_3, \quad \psi_x = \omega\Phi_4, \quad \psi_y = \omega\Phi_5, \tag{17}$$

where $\omega = 0$ is equation of the border of the shell planform. In order to realize the solution structure (17) we should construct the equation of the border $\omega = 0$. Using the R-operations $\wedge_0, \vee_0$ [13], we build the equation in the form:

$$\omega = (f_1 \vee_0 f_2) \wedge_0 f_3, \tag{18}$$

where f_i $(i = 1,2,3)$ are defined as

$$f_1 = \left(\left(x^2 - a_1{}^2\right)/2a_1\right) \geq 0, \quad f_2 = \left(\left(b_1{}^2 - y^2\right)/2b_1\right) \geq 0, \quad f_3 = \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \geq 0.$$

Investigation results for FG clamped spherical shell with planform shown in Fig.3 have been fulfilled. Calculations have been carried out for the following values of the geometric parameters:

1) $b_1/2a = 0.175;\ a_1/2a = 0.15$ **;** **2)** $b_1/2a = 0.245; a_1/2a = 0.15 \quad (b_1/b \to 1)$.

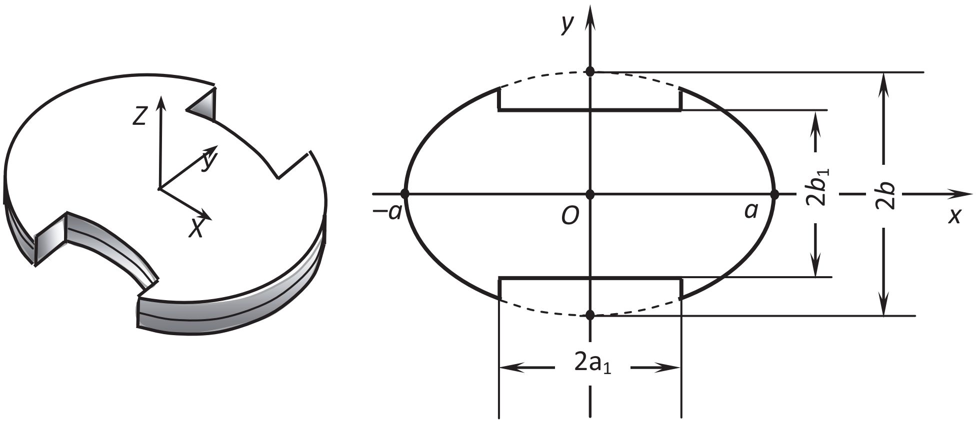
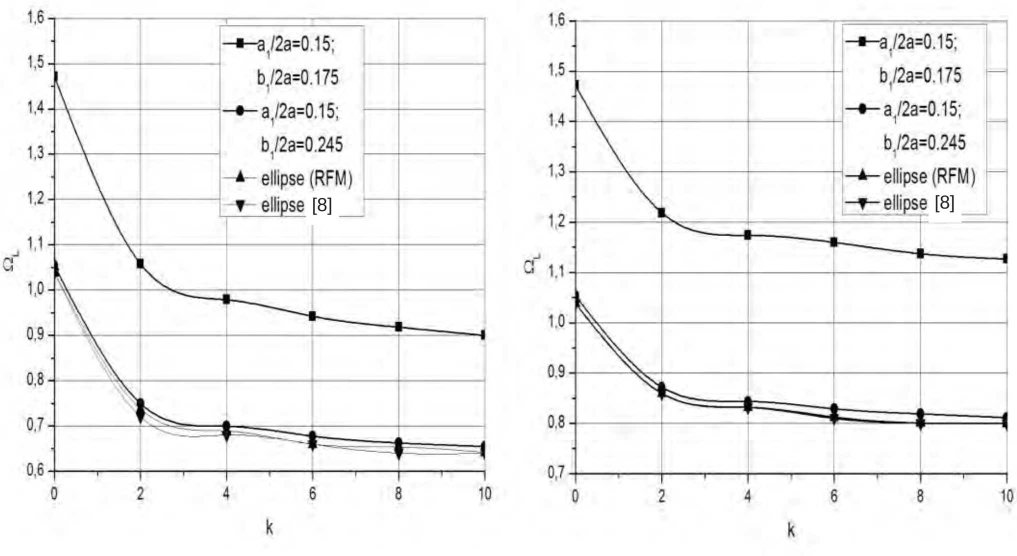


Fig. 3. Spherical FG shallow shell and its planform

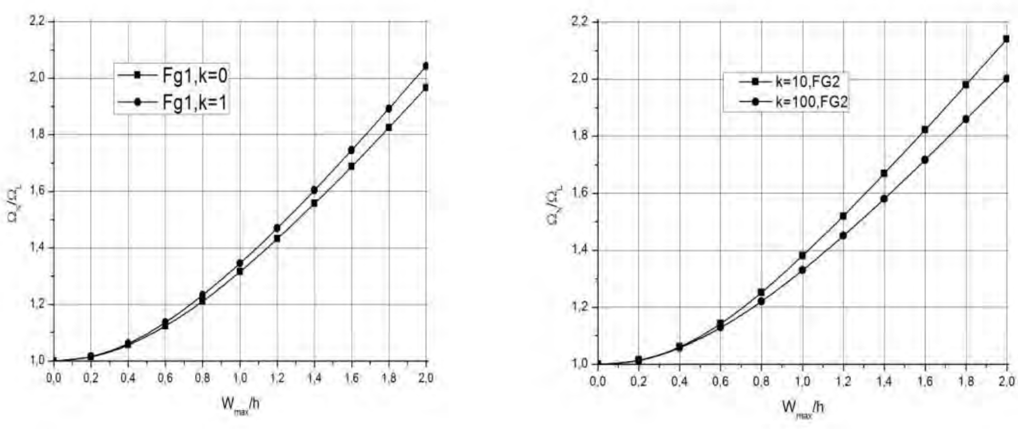
Effect of the volume fraction exponent k on linear frequencies $\Omega_L = \lambda_1 a^2 h \sqrt{\rho_c / E_c}$ of FG shells of the materials FG1 and FG2 is shown in Fig 4a and Fig.4b respectively.



a) b)

Fig.4. Influence of the power-law exponent *k* on the frequency parameter Ω_L of the FG shallow shell with complex planform (a - material FG1, b – material FG2)

In the second case 2) the geometric shape of the plan (Fig. 3) goes to the elliptical and the results can be compared with the results of work [8]. These graphs show that they are practically the same, which confirms the accuracy of the solution of the linear problem in the case of complex geometry.



a) b)

Fig.5. Backbone curves (*a*-material FG1, *b*-material FG2)

Backbone curves of the FG spherical shells made of material FG1 are presented in Fig.5a, but for the shell made of material FG2 are shown in Fig 5b. Geometrical parameters coincide with parameters chosen for linear problem: $b_1/2a = 0.175$; $a_1/2a = 0.15$. In the both cases the backbone curves have a strong hardening behavior, monotonically increasing, which is typical for moderately thick clamped shells ($h/2a = 0.1$).

It was established that if the ratio $b_1/2a \to 0.25$, i.e. $b_1/b \to 1$, then the backbone curves approaches the corresponding curves for shells with elliptical planform, which confirms the validation of the presented results.

Conclusions

This paper proposes the method of investigation of geometrically nonlinear free oscillations of functionally graded shallow shells with complex plan form. This method is based on the theory of R-functions, Ritz's variational method, procedures by Bubnov-Galerkin and Runge-Kutta methods. To clarify the theory of shallow shells of the 1-st order proposed approach is implemented within the POLE-RL. The conducted tests for shells that are based on the square and elliptical planform proves the reliability and effectiveness of the proposed method, an illustration of which is made for shells with a complicated shape of the plan. In the future the developed method is planned to be used not only for the boundary conditions corresponding to the clamped FG shells, but for others, including mixed. In addition it would be interesting to apply the approach developed for the mathematical formulation of the problem in mixed form and not only in the movements.

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