Asymptotic Analysis and Limiting Phase Trajectories in the Dynamics of Spring Pendulum

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Abstract Spring pendulum is a widely discussed two degree-of-freedom (DOF) mechanical systems in numerous references. In this paper the asymptotic approach and limiting phase trajectories (LPT) have been applied to analyze the two DOF mathematical model of a spring pendulum. The LPT and multiple timescale (MTS) methods are effective tools of the investigation of non-linear systems. Some interesting and important aspects of dynamics of the system are discussed. The main attention is focused on the non-steady-state vibrations when the energy is intensively exchanged. Then with increasing values of the selected parameters, a sudden change in the character of vibrations is observed. These phenomena are very well described by the LPT. The method allows to determine the critical values of the parameters responsible for the mentioned transitions. Our analytical studies are verified by numerical calculations.

1 Introduction

The steady-state vibrations are, in general, mainly observed in engineering practice. However, in some cases of sharp resonance, the transient stage of the oscillatory process and its relaxation can last a long time. Energy exchange and non-stationary processes appear in many dynamical systems and they are of great interest of many researchers. This problem has been widely discussed in [2, 6]. It is usually studied numerically due to occurred essential mathematical difficulties [2]. However, in recent years, one may observe a great interest in successful application of modern asymptotic methods to engineering-oriented problems [1, 4]. In particular, a novel

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idea for an effective study of non-linear dynamical systems is linked with a concept of the so-called limiting phase trajectories (LPT) (see [3]).

The analysis of the non-linear spring pendulum is carried out in the paper. The unsteady-state oscillations near resonance are discussed. The pendulum-type mechanical systems with non-linear and parametric interactions exhibit a rich behaviour, and hence their understanding and prediction are important both from a point of view of the theory and application. Pendulums are relatively simple systems; neverthelessthey can be used to simulate the dynamics of a wide variety of engineering devices and machine parts. The coupling of the equations of motion causes possibility of autoparametric excitation and is connected to the energy exchange between modes of vibrations [8]. The energy transfer is well known in dynamics of multi-degree-of-freedom systems and is widely discussed by many authors [5, 7]. A key role either for theoretical- or application-oriented analysis is played by prediction and determination of thresholds (critical set of parameters), where transitions of system dynamics take place from a periodic quasi-linear to strongly non-linear behaviour. It can be observed in the neighbourhood of a resonance. Such critical value of non-linear parameter of the spring pendulum is determined in the paper.

2 Formulation of the Problem

Let us consider the planar motion of a mass attached to the massless non-linear spring. The examined system is shown in Fig. 1.

The Lagrangian of the system is given by

$$\mathsf{L} = mg\cos\phi\,(L_0 + Z) - \frac{1}{2}k_1 Z^2 - \frac{1}{4}k_2 Z^4 + \frac{1}{2}m\left(\dot{Z}^2 + (L_0 + Z)^2\dot{\phi}^2\right) \quad (1)$$

where m is the mass of the pendulum, L_0 is the length of the nonstretched spring, k_1 and k_2 are the stiffness coefficients, g is the Earth's acceleration and $Z(t)$ and $\phi(t)$ are generalized coordinates (see Fig. 1). The magnitudes of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on the mass along and transversally to the pendulum are $F_1(t) = F_1\cos(\Omega_1\ t)$ and $F_2(t) = F_2\cos(\Omega_2 t)$. Forces of linear viscous damping are considered to be present in both longitudinal and swing motions of the pendulum (C_1 and C_2 are viscous coefficients).

The equations of motion have been obtained using Lagrange equations of the second type. Their non-dimensional form follows

$$\ddot{z}+c_1\dot{z}+z+\alpha\, z^3 + 3\alpha\, z_r^2 z + 3\alpha z_r z^2 + w^2\,(1-\cos\varphi) - (z+1)\,\dot{\varphi}^2 = f_1\cos\left(p_1\tau\right) \quad (2)$$

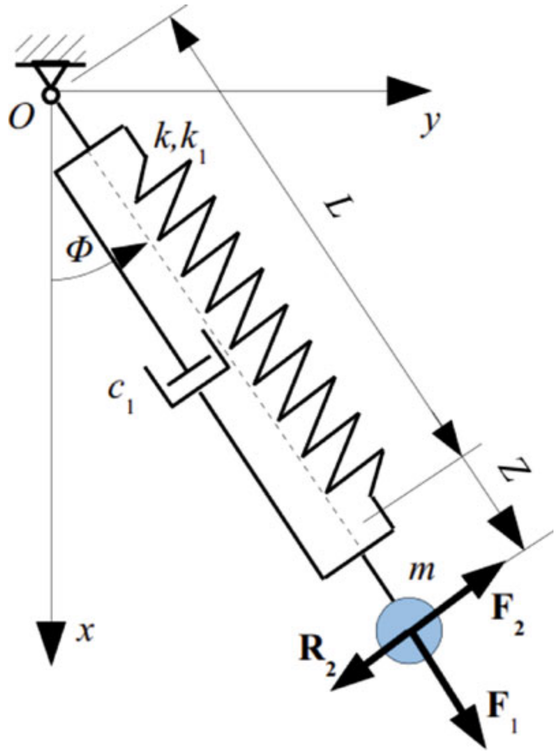

Fig. 1 The spring pendulum

$$(z+1)\left((z+1)\ddot{\varphi}+w^2\sin\varphi+\dot{\varphi}\left(c_2+2\dot{z}\right)\right)=(z+1)\,f_2\cos\left(p_2\tau\right) \tag{3}$$

where $z = Z/L$, $L = L_0 + Z_r$, $c_1 = C_1/m\omega_1$, $c_2 = C_2/L^2m\omega_1$, $w = \omega_2/\omega_1$, $\omega_2 = \sqrt{g/L}$, $\omega_1 = \sqrt{k_1/m}$, $p_1 = \Omega_1/\omega_1$, $p_2 = \Omega_2/\omega_1$, $f_1 = F_1/Lm\omega_1^2$, $f_2 = F_2/Lm\omega_1^2$ and dimensionless time $\tau = t\omega_1$. Now z and ϕ are functions of τ, whereas z_r denotes the elongation of the spring at the static equilibrium position and fulfils the equation

$$\alpha\, z_r^3 + z_r = w^2 \tag{4}$$

The second equation of motion (3) gives, among others, a trivial solution $z = -1$ which has no physical meaning and should be rejected.

Vibrations of the system are investigated in the neighbourhood of the equilibrium position; hence, the trigonometric functions can be substituted by their power series approximations

$$\sin\varphi \approx \varphi - \varphi^3/6, \quad \cos\varphi \approx 1 - \varphi^2/2 \tag{5}$$

which limit the angle about to π/6 with precision of four significant digits.

The above remarks lead to a new form of the equations of motion

$$\ddot{z} + c_1\dot{z} + z + \alpha\, z^3 + 3\alpha\, z_r^2 z + 3\alpha\, z_r z^2 + \frac{1}{2}w^2\varphi^2 - (z+1)\,\dot{\varphi}^2 = f_1 \cos(p_1\tau) \tag{6}$$

$$(z+1)\,\ddot{\varphi} + w^2\left(\varphi - \frac{\varphi^3}{6}\right) + \dot{\varphi}\,(c_2 + 2\dot{z}) = f_2 \cos(p_2\tau) \tag{7}$$

Let us assume the homogeneous initial conditions

$$z(0) = 0,\;\; \dot{z}(0) = 0,\;\; \varphi(0) = 0,\;\; \dot{\varphi}(0) = 0 \tag{8}$$

The above initial problem described by the coupled and non-linear equations is investigated.

2.1 Complex Representation of the Problem

Let us introduce the phase space coordinates $\dot{z}(\tau) = v(\tau)$ and $\dot{\varphi}(\tau) = \beta(\tau)$ into (6)–(8) and then rewrite the problem in the form

$$\dot{v} + c_1 v + z + \alpha\, z^3 + 3\alpha\, z_r^2 z + 3\alpha z_r z^2 + \frac{1}{2}w^2\varphi^2 - (z+1)\,\beta^2 = f_1 \cos(p_1\tau)\,, \tag{9}$$

$$(z+1)\,\dot{\beta} + w^2\left(\varphi - \frac{\varphi^3}{6}\right) + \beta\,(c_2 + 2\varpi) = f_2 \cos(p_2\tau)\,, \tag{10}$$

$$z(0) = 0,\;\; v(0) = 0,\;\; \varphi(0) = 0,\;\; \beta(0) = 0. \tag{11}$$

Then, the approach proposed in the paper [3] is applied. Introduction of the complex-valued functions

$$\Psi_z = v + i\; z,\quad \Psi_\varphi = \beta + i\; w\,\varphi,\;\; \overline{\Psi}_z = v - i\; z,\quad \overline{\Psi}_\varphi = \beta - i\; w\,\varphi \tag{12}$$

converts the problems (9)–(11) to the complex form

$$\begin{aligned} &\tfrac{1}{2}\left(\dot{\Psi}_z + \dot{\overline{\Psi}}_z\right) + \tfrac{c_1}{2}\left(\Psi_z + \overline{\Psi}_z\right) - \tfrac{1}{2}i\,\left(\Psi_z - \overline{\Psi}_z\right) + \tfrac{1}{8}i\;\alpha\left(\Psi_z - \overline{\Psi}_z\right)^3 \\ &- \tfrac{3}{2}\alpha\; z_r^2\left(\Psi_z - \overline{\Psi}_z\right) - \tfrac{3}{4}\alpha\; z_r\left(\Psi_z - \overline{\Psi}_z\right)^2 - \tfrac{1}{8}\left(\Psi_\varphi - \overline{\Psi}_\varphi\right)^2 \\ &+ \tfrac{1}{4}\left(\Psi_\varphi + \overline{\Psi}_\varphi\right)^2\left(\tfrac{1}{2}i\,\left(\Psi_z - \overline{\Psi}_z\right) - 1\right) = f_1 \cos(p_1\tau)\,, \end{aligned} \tag{13}$$

$$\begin{aligned}&\tfrac{1}{2}\left(1-\tfrac{1}{2}i\left(\Psi_z-\overline{\Psi}_z\right)\right)\left(\dot{\Psi}_\varphi+\dot{\overline{\Psi}}_\varphi\right)+\tfrac{1}{2}\left(\Psi_\varphi+\overline{\Psi}_\varphi\right)\left(c_2+\left(\Psi_z+\overline{\Psi}_z\right)\right)\\&+w^2\left(-\tfrac{i\left(\Psi_\varphi-\overline{\Psi}_\varphi\right)^3}{48w^3}-\tfrac{i\left(\Psi_\varphi-\overline{\Psi}_\varphi\right)}{2w}\right)=f_2\cos\left(p_2\tau\right),\end{aligned}\tag{14}$$

$$\Psi_z(0)=0,\;\;\Psi_\varphi(0)=0,\overline{\Psi}_z(0)=0,\;\;\overline{\Psi}_\varphi(0)=0\tag{15}$$

The complex conjugate equations similar to (13) and (14) are also derived. They and all the consequent formulas are not written for greater clarity.

Afterwards the exponential form of the functions $\Psi_z(\tau)=\psi_z(\tau)e^{i\tau}$ and $\Psi_\varphi(\tau)=w\psi_\varphi(\tau)e^{iw\tau}$ is postulated which leads to the new form of the governing equations

$$\begin{aligned}&\dot{\psi}_z+\tfrac{1}{8}i\;e^{-4i\tau}\alpha\left(e^{2i\tau}\psi_z-\overline{\psi}_z\right)^3+\tfrac{1}{2}c_1\left(\psi_z+\overline{\psi}_ze^{-2i\tau}\right)-\tfrac{3}{2}i\alpha\;z_r^2\left(\psi_z-\overline{\psi}_ze^{2i\tau}\right)\\&-\tfrac{3}{4}\alpha\;z_re^{-3i\tau}\left(\overline{\psi}_z-e^{2i\tau}\psi_z\right)^2-\tfrac{3}{8}w^2e^{-i\tau(1+2w)}\left(\overline{\psi}_\varphi^2+\psi_\varphi^2e^{4i\tau w}\right)\\&+\tfrac{1}{4}ie^{2i\tau}w^2\psi_\varphi\overline{\psi}_\varphi\left(e^{2i\tau}\psi_z-\overline{\psi}_z+e^{i\tau}\right)\\&+\tfrac{1}{8}ie^{-2i\tau(1+w)}w^2\left(e^{2i\tau}\psi_z-\overline{\psi}_z\right)\left(e^{4iw\tau}\psi_\varphi^2+\overline{\psi}_\varphi^2\right)=f_1e^{-i\tau}\cos\left(p_1\tau\right),\end{aligned}\tag{16}$$

$$\begin{aligned}&w\dot{\psi}_\varphi+\tfrac{1}{2}ie^{-i\tau}w\dot{\psi}_\varphi\left(\overline{\psi}_z-e^{2i\tau}\psi_z\right)-\tfrac{1}{2}c_2w\left(\psi_\varphi-e^{-2iw\tau}\overline{\psi}_\varphi\right)+\tfrac{1}{4}e^{i\tau}w\left(w+2\right)\psi_\varphi\psi_z\\&-\tfrac{1}{4}e^{i\tau(1-2w)}w\left(w-2\right)\overline{\psi}_\varphi\psi_z-\tfrac{1}{4}e^{-i\tau}w\left(w-2\right)\psi_\varphi\overline{\psi}_z+\tfrac{1}{4}e^{-i\tau(1+2w)}w\left(w+2\right)\overline{\psi}_\varphi\overline{\psi}_z\\&-\tfrac{1}{48}iw^2e^{-4i\tau w}\left(\psi_\varphi e^{2i\tau w}-\overline{\psi}_\varphi\right)^3=f_2e^{-i\tau w}\cos\left(p_2\tau\right),\end{aligned}\tag{17}$$

with the initial conditions

$$\psi_z(0)=0,\;\;\psi_\varphi(0)=0,\overline{\psi}_z(0)=0,\;\;\overline{\psi}_\varphi(0)=0\tag{18}$$

3 Asymptotic Solution

The problems (16)–(18) can be efficiently solved by the asymptotic multiple scale method. The assumptions of smallness of the parameters are proposed in the form

$$c_1=\tilde{c}_1\varepsilon^2,\;\;c_2=\tilde{c}_2\varepsilon^2,\;\;z_r=\tilde{z}_r\varepsilon,\;\;f_1=\tilde{f}_1\varepsilon^3\;\;f_2=\tilde{f}_2\varepsilon^3,\tag{19}$$

where ε is the so-called small parameter.

Adopting three timescales in the analysis the solutions are searched in the following form of series with respect to the small parameter:

$$\psi_z(\tau;\varepsilon) = \sum_{k=1}^{k=3} \varepsilon^k \xi_{zk}(\tau_0,\tau_1,\tau_2) + O\left(\varepsilon^4\right),$$
$$\psi_\varphi(\tau;\varepsilon) = \sum_{k=1}^{k=3} \varepsilon^k \xi_{\varphi k}(\tau_0,\tau_1,\tau_2) + O\left(\varepsilon^4\right), \tag{20}$$

and the differential operator has the form

$$\frac{d}{d\tau} = \frac{\partial}{\partial\tau_0} + \varepsilon\frac{\partial}{\partial\tau_1} + \varepsilon^2\frac{\partial}{\partial\tau_2} + ... \tag{21}$$

3.1 Motion Near Resonance

Let us focus attention on the case of main resonance $p_2 \approx w$ and $p_1 \approx 1$. In order to deal this case the following substitutions have been done

$$p_1 = 1 + \sigma_1 \quad \text{and} \quad p_2 = w + \sigma_2, \tag{22}$$

where $\sigma_1 = \tilde{\sigma}_1\varepsilon^2$ and $\sigma_2 = \tilde{\sigma}_2\varepsilon^2$ are detuning parameters.

Introducing now (19), (20) and (22) into (16) and (17) and replacing the ordinary derivatives by the differential operator (21) we obtain two equations in which the small parameter ε appears. These equations should be satisfied for any value of the small parameter, so after sorting them with respect to the powers of ε we get

(i) the equations of order ε^1

$$\frac{\partial\xi_{z1}}{\partial\tau_0} = 0, \tag{23}$$

$$\frac{\partial\xi_{\varphi1}}{\partial\tau_0} = 0, \tag{24}$$

(ii) the equations of order ε^2

$$\frac{\partial\xi_{z2}}{\partial\tau_0} + \frac{\partial\xi_{z1}}{\partial\tau_1} - \frac{1}{4}w^2 e^{-i\tau_0}\xi_{\varphi1}\overline{\xi}_{\varphi1} - \frac{3}{8}w^2 e^{-i\tau_0(1+2w)}\left(e^{4i\tau_0 w}\xi_{\varphi1}^2 - \overline{\xi}_{\varphi1}^2\right) = 0, \tag{25}$$

$$\begin{aligned} & w\tfrac{\partial\xi_{\varphi2}}{\partial\tau_0} - \tfrac{1}{2}ie^{i\tau_0}w\tfrac{\partial\xi_{\varphi1}}{\partial\tau_0}\xi_{z1} + \tfrac{1}{2}ie^{-i\tau_0}w\tfrac{\partial\xi_{\varphi1}}{\partial\tau_0}\overline{\xi}_{z1} + w\tfrac{\partial\xi_{\varphi1}}{\partial\tau_1} + \tfrac{1}{4}e^{i\tau_0}w\xi_{\varphi1}\xi_{z1}(w+2) \\ & - \tfrac{1}{4}e^{i\tau_0(1-2w)}w^2\overline{\xi}_{\varphi1}\xi_{z1} - \tfrac{1}{4}e^{-i\tau_0}w\xi_{\varphi1}\overline{\xi}_{z1}(w-2) + \tfrac{1}{4}e^{-i\tau_0(1+2w)}w^2\overline{\xi}_{\varphi1}\overline{\xi}_{z1} \\ & + \tfrac{1}{2}e^{i\tau_0(1-2w)}w\overline{\xi}_{\varphi1}\xi_{z1} + \tfrac{1}{2}e^{-i\tau_0(1+2w)}w\overline{\xi}_{\varphi1}\overline{\xi}_{z1} = 0, \end{aligned} \tag{26}$$

(iii) the equations of order ε^3

$$\frac{\partial\xi_{z3}}{\partial\tau_0}+\frac{\partial\xi_{z2}}{\partial\tau_1}+\frac{\partial\xi_{z1}}{\partial\tau_2}-\frac{3}{2}i\tilde{\alpha}\,z_r^2\left(\xi_{z1}-\overline{\xi}_{z1}e^{-2i\tau_0}\right)-\frac{3}{4}\tilde{\alpha}\,z_re^{-3i\tau_0}\left(\overline{\xi}_{z1}-\xi_{z1}e^{2i\tau_0}\right)^2$$
$$+\frac{1}{8}i\tilde{\alpha}e^{-4i\tau_0}\,z_r^2\left(\xi_{z1}e^{2i\tau_0}-\overline{\xi}_{z1}\right)^3+\frac{\tilde{c}_1}{2}\left(\overline{\xi}_{z1}e^{-2i\tau_0}+\xi_{z1}\right)-\frac{1}{2}\tilde{f}_1e^{i\tilde{\sigma}_1\tau_2}$$
$$-\frac{1}{4}w^2e^{-i\tau_0}\left(\xi_{\varphi2}\overline{\xi}_{\varphi1}+\overline{\xi}_{\varphi2}\xi_{\varphi1}\right)-\frac{1}{4}w^2\left|\xi_{\varphi1}\right|^2\left(\xi_{z1}-\overline{\xi}_{z1}e^{-2i\tau_0}\right)$$
$$-\frac{3}{4}w^2\left(\xi_{\varphi1}\xi_{\varphi2}e^{i\tau_0(2w-1)}+\overline{\xi}_{\varphi2}\xi_{\varphi1}e^{-i\tau_0(2w+1)}\right)-\frac{1}{2}\tilde{f}_1e^{-i\tilde{\sigma}_1\tau_0-i\tilde{\sigma}_1\tau_2}$$
$$+\frac{1}{8}iw^2\left(\xi_{\varphi1}^2\xi_{z1}e^{2i\tau_0w}+\overline{\xi}_{\varphi1}^2\xi_{z1}e^{+2i\tau_0w}+\xi_{\varphi1}^2\overline{\xi}_{z1}e^{2i\tau_0(w-1)}+\overline{\xi}_{\varphi1}^2\overline{\xi}_{z1}e^{2i\tau_0(w+1)}\right)=0, \tag{27}$$

$$\frac{\partial\xi_{\varphi3}}{\partial\tau_0}-\frac{1}{2}iwe^{-i\tau_0}\left[\left(\xi_{z1}e^{2i\tau_0}-\overline{\xi}_{z1}\right)\left(\frac{\partial\xi_{\varphi2}}{\partial\tau_0}-\frac{\partial\xi_{\varphi1}}{\partial\tau_1}\right)-\left(\xi_{z2}e^{2i\tau_0}-\overline{\xi}_{z2}\right)\frac{\partial\xi_{\varphi1}}{\partial\tau_0}\right]$$
$$+w\frac{\partial\xi_{\varphi2}}{\partial\tau_1}+w\frac{\partial\xi_{\varphi1}}{\partial\tau_2}+\frac{\tilde{c}_2}{2}w\left(\xi_{\varphi1}+\overline{\xi}_{\varphi1}e^{-2iw\tau_0}\right)-\frac{1}{2}\tilde{f}_2e^{i\tilde{\sigma}_2\tau_2}-\frac{1}{2}\tilde{f}_2e^{-i(2w\tau_0+\tilde{\sigma}_2\tau_2)}$$
$$+\frac{1}{4}e^{i\tau_0}w\left(2+w\right)\left(\xi_{z2}\xi_{\varphi1}+\xi_{z1}\xi_{\varphi2}\right)-\frac{1}{4}e^{-i\tau_0}w\left(w-2\right)\left(\overline{\xi}_{z2}\xi_{\varphi1}+\overline{\xi}_{z1}\xi_{\varphi2}\right)$$
$$-\frac{1}{48}iw^2e^{-4iw\tau_0}\left(\xi_{\varphi1}e^{2iw\tau_0}-\overline{\xi}_{\varphi1}\right)^3-\frac{1}{4}w\left(w-2\right)e^{i\tau_0(1-2w)}\xi_{z2}\overline{\xi}_{\varphi1}$$
$$+\frac{1}{4}w\left(w+2\right)e^{-i\tau_0(1+2w)}\overline{\xi}_{z2}\overline{\xi}_{\varphi1}+\frac{1}{2}we^{i\tau_0(1-2w)}\xi_{z1}\overline{\xi}_{\varphi2}\left(1-\frac{w}{2}\right)$$
$$+\frac{1}{2}we^{-i\tau_0(1+2w)}\overline{\xi}_{z1}\overline{\xi}_{\varphi2}\left(1+\frac{w}{2}\right)=0. \tag{28}$$

The requirement of zeroing of secular terms in (23)–(26) causes the functions $\xi_{z1}\left(\tau_2\right),\overline{\xi}_{z1}\left(\tau_2\right),\xi_{\varphi1}\left(\tau_2\right),\overline{\xi}_{\varphi1}\left(\tau_2\right)$ to depend only on the slowest timescale τ_2.

Solutions of the second-order equations (25) and (26)

$$\xi_{z2}=G_1\left(\tau_1,\tau_2\right)+\frac{3iw^2}{8\left(1-2w\right)}\left(e^{i\tau_0(2w-1)}\xi_{\varphi1}^2+e^{-i\tau_0(2w+1)}\overline{\xi}_{\varphi1}^2\right)+\frac{1}{4}iw^2e^{-i\tau_0}\xi_{\varphi1}\overline{\xi}_{\varphi1}, \tag{29}$$

$$\xi_{\varphi2}=G_2\left(\tau_1,\tau_2\right)+\frac{1}{4}ie^{i\tau_0}\left(2+w\right)\xi_{z1}\xi_{\varphi1}+\frac{1}{4}ie^{-i\tau_0}\left(w-2\right)\overline{\xi}_{z1}\xi_{\varphi1}$$
$$+\frac{i\left(w-2\right)e^{i\tau_0(1-2w)}\xi_{z1}\overline{\xi}_{\varphi1}}{8w-4}-\frac{i\left(w+2\right)e^{-i\tau_0(1+2w)}\overline{\xi}_{z1}\overline{\xi}_{\varphi1}}{8w+4}, \tag{30}$$

are then introduced into equations of the third order (27) and (28). According to the initial conditions (18), $G_1=0$ and $G_2=0$.

Assuming that the system vibrates far from the internal resonance $2w-1=0$, the requirement that the solutions should be limited in time leads to the equations

$$\frac{\partial\xi_{z1}}{\partial\tau_2}+\frac{\tilde{c}_1}{2}\xi_{z1}-\frac{3}{2}i\;\tilde{\alpha}\;\tilde{z}_r^2\xi_{z1}p-\frac{3}{8}i\;\tilde{\alpha}\left|\xi_{z1}\right|\;\xi_{z1}+\frac{3i\;w^2\left(w^2-1\right)}{4-16w^2}\left|\xi_{\varphi1}\right|\;\xi_{z1}=\frac{1}{2}\tilde{f}_1e^{i\tau_2\tilde{\sigma}_1}, \tag{31}$$

$$w\frac{\partial \xi_{\varphi 1}}{\partial \tau_2} + \frac{\tilde{c}_2}{2} w \xi_{\varphi 1} - \frac{1}{4} i\ w^2 \left|\xi_{z1}\right|\ \xi_{\varphi 1} + \frac{i\ w^2\left(w^2+2\right)}{16w^2-4} \left|\xi_{z1}\right|\ \xi_{\varphi 1} + \frac{i\ w^2\left(8w^4-7w^2-1\right)}{64w^2-16} \left|\xi_{\varphi 1}\right|\ \xi_{\varphi 1} = \frac{1}{2}\tilde{f}_2 e^{i\tau_2\tilde{\sigma}_2}. \tag{32}$$

Now real representation of the functions ξ_{z1} and $\xi_{\varphi 1}$ of the following form

$$\xi_{z1}\left(\tau_2\right) = \tilde{a}_1\left(\tau_2\right) e^{i\delta_1(\tau_2)},\ \ \xi_{\varphi 1}\left(\tau_2\right) = \tilde{a}_2\left(\tau_2\right) e^{i\delta_2(\tau_2)}, \quad \tilde{a}_i = a_i\varepsilon \quad \text{for} \quad i = 1, 2 \tag{33}$$

is introduced to the above secular terms (31) and (32).

Then we go back to the original denotations according to (19) and take advantage of the definition (21). Comparison of the real and imaginary parts of both sides of (31) and (32) leads to four modulation equations with respect to amplitudes a_1, a_2 and modified phases θ_1, θ_2:

$$\frac{da_1}{d\tau} = -\frac{1}{2}c_1 a_1 + \frac{1}{2} f_1 \cos\theta_1, \tag{34}$$

$$a_1\frac{d\theta_1}{d\tau} = -\frac{3}{2}z_r^2\alpha\ a_1 + \sigma_1 a_1 - \frac{3}{8}\alpha\ a_1^3 + \frac{3w^2\left(w^2-1\right)}{4-16w^2}a_1 a_2^2 - \frac{1}{2} f_1 \sin\theta_1, \tag{35}$$

$$\frac{da_2}{d\tau} = -\frac{1}{2}c_2 a_2 + \frac{1}{2w} f_2 \cos\theta_2, \tag{36}$$

$$a_2\frac{d\theta_2}{d\tau} = \sigma_2 a_2 + \frac{3w\left(w^2-1\right)}{4-16w^2}a_2 a_1^2 + \frac{w\left(8w^4-7w^2-1\right)}{64w^2-16}a_2^3 - \frac{1}{2w} f_2 \sin\theta_2, \tag{37}$$

where modified phases θ_1, θ_2 are defined as follows:

$$\delta_1\left(\tau_2\right) = \tau_2\sigma_1 - \theta_1\left(\tau_2\right), \quad \delta_2\left(\tau_2\right) = \tau_2\sigma_2 - \theta_2\left(\tau_2\right). \tag{38}$$

The above definitions cause the systems (34)–(37) to become an autonomous one. It describes the dynamics of the non-linear spring pendulum near simultaneously occurring main resonances.

4 Examples

The LPT concept allows to describe the intensive energy exchange between the degrees of freedom and external sources. The system examined in the paper is especially sensitive to changes of the value of the parameter α responsible for non-linear characteristics of the spring. One can observe a critical value $\alpha = \alpha_{LPT}$ for which the character of the vibrations dramatically changes. The value of α_{LPT}

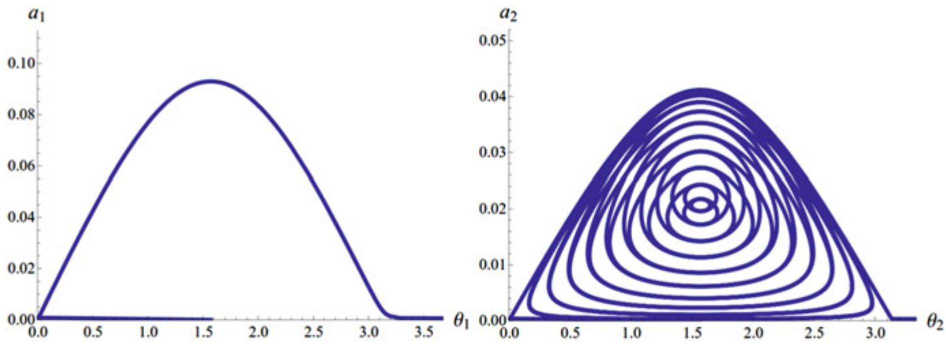

Fig. 2 Limiting phase trajectories for longitudinal and swing vibrations; $\alpha = 0.3 < \alpha_{LPT}$

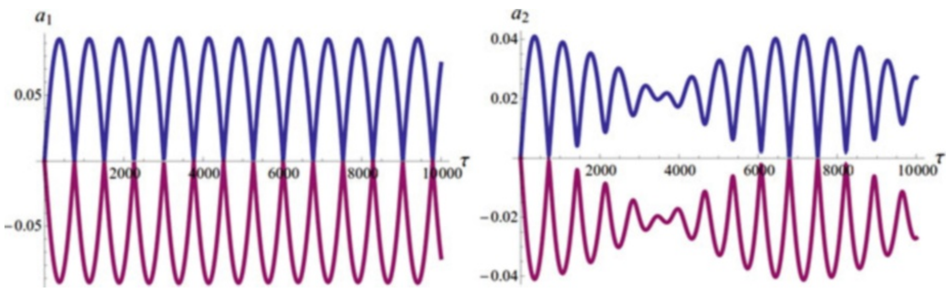

Fig. 3 Amplitudes modulations; $\alpha = 0.3 < \alpha_{LPT}$

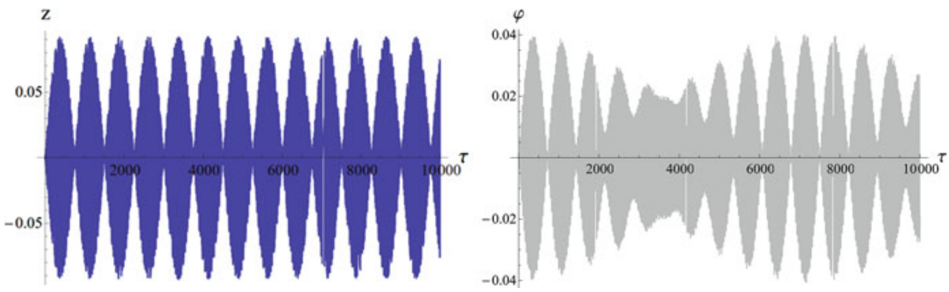

Fig. 4 Time histories obtained numerically; $\alpha = 0.3 < \alpha_{LPT}$

depends on all the parameters of the system. In the case of certain one degree-of-freedom (DOF) systems α_{LPT} can be obtained analytically [2, 3]. When the number of DOF is higher than one and couplings appear in the equations of modulation, α_{LPT} can be obtained approximately.

The results of calculations for the chosen values of parameters $\sigma_1 = 0.01$, $\sigma_2 = 0.01, f_1 = 0.0008, f_2 = 0.00008$, $c_1 = 0, c_2 = 0$, $w = 0.21$ are presented below. For these parameters $\alpha_{LPT} \approx 0.654$. Figures 2, 3 and 4 show some graphs concerning the case when $\alpha = 0.3 < \alpha_{LPT}$.

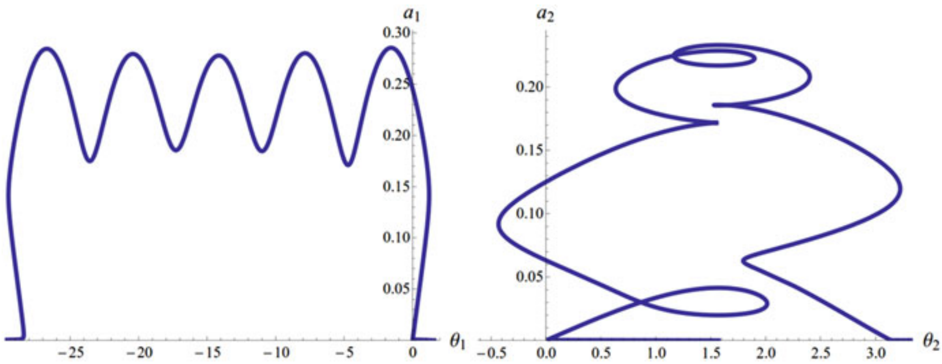

Fig. 5 Limiting phase trajectories for spring and swing vibrations; $\alpha = 0.7 > \alpha_{LPT}$

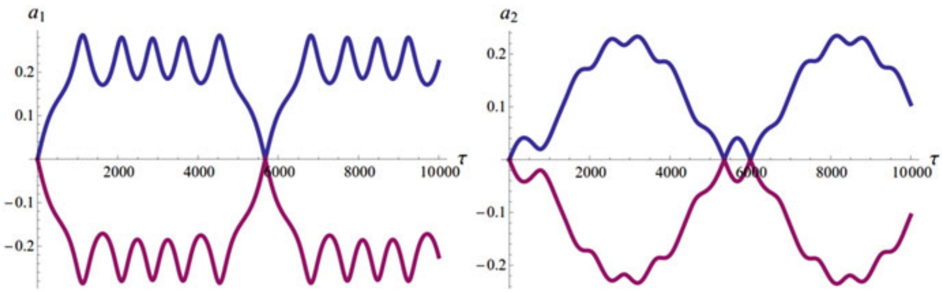

Fig. 6 Amplitudes modulations; $\alpha = 0.7 > \alpha_{LPT}$

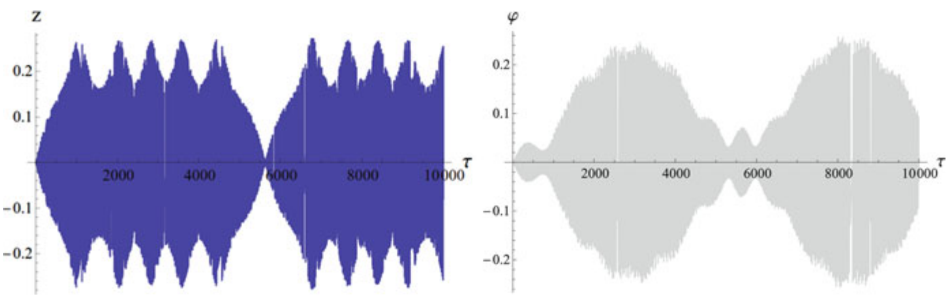

Fig. 7 Time histories obtained numerically; $\alpha = 0.7 > \alpha_{LPT}$

In Figs. 2, 3 and 4 the intensive energy exchange between the system and its surrounding can be observed. The amplitudes are relatively small and the vibrations are quasi-linear. The time histories presented in Fig. 4 have been received by numerical solution to the problem (6)–(8). The amplitude modulations and time histories (compare Figs. 3 and 4) are highly compatible.

For $\alpha > \alpha_{LPT}$ some non-linear effects occur. In Figs. 5, 6 and 7 amplitudes are much larger than in the case when $\alpha < \alpha_{LPT}$ and the vibrations become strongly non-linear.

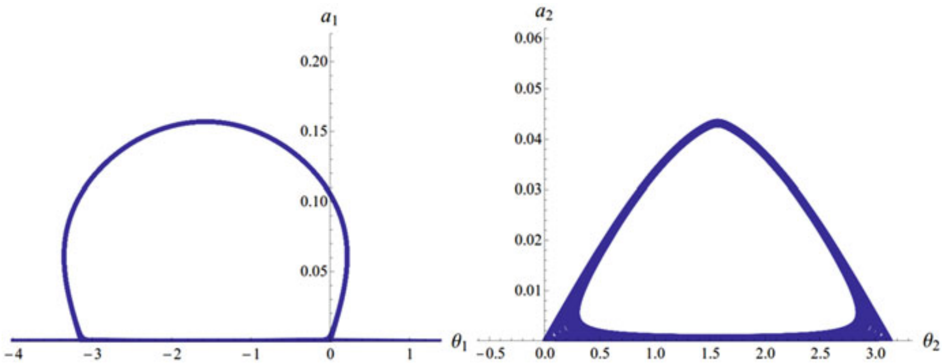

Fig. 8 Limiting phase trajectories for spring and swing vibrations; $\alpha = 2.0 >> \alpha_{LPT}$

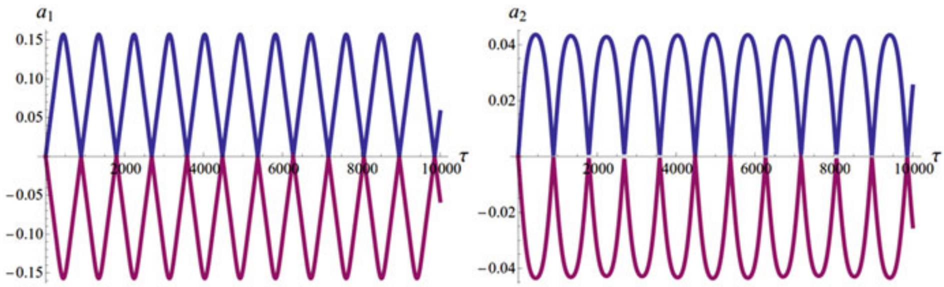

Fig. 9 Amplitudes modulations; $\alpha = 2.0 >> \alpha_{LPT}$

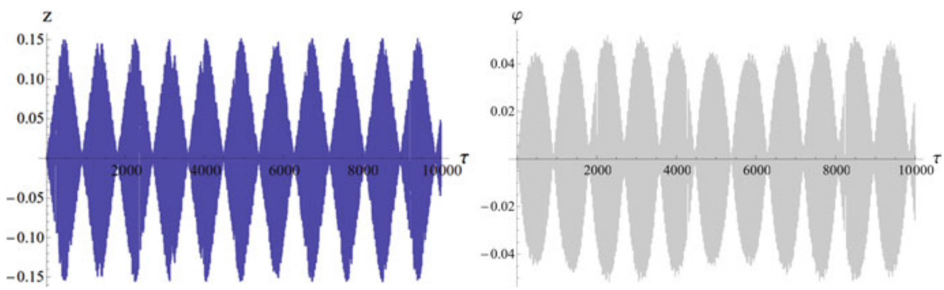

Fig. 10 Amplitudes modulations; $\alpha = 2.0 >> \alpha_{LPT}$

In the case $\alpha >> \alpha_{LPT}$ modulations of amplitudes of both coordinates become again more regular, as is shown in Figs. 9 and 10. However, their shape indicates the non-linear effects. The longitudinal vibration tends to sawtooth form with the increase of non-linearity parameter. A synchronization between the amplitude modulations of both general coordinates in the slow timescale is observable (Fig. 9). The oscillations of the system tend to steady state what is seen not only in the time history but also in the phase – amplitude plane (Fig. 8).

5 Conclusions

Analytical study of the non-linear spring pendulum in planar motion has been carried out. After transformation of the governing equations of motion to the complex representation, the asymptotic analysis with the help of multiple timescale (MTS) method has been applied. That approach leads to obtain a set of differential equations of simpler form than the original ones. It is worth to note that, thanks to the application of the MTS variant with three timescales, the non-linear terms as well as the most important coupled terms between the generalized coordinates have been preserved in the equations of the simplified mathematical model. The performed investigations have been focused on the nonsteady vibrations of the forced system near the simultaneously occurring external resonances. The modulation equations concerning this case have been derived from the equations of first order as well as from the requirement of vanishing the secular terms of the equations of higher order. The solutions obtained analytically from the equations of modulation of amplitudes and phases have been verified by comparing them with the solutions which are received numerically from the original equations of motion. Their high accuracy has been confirmed in all performed numerical simulations.

The main advantage of the asymptotic solutions consists in achieving qualitative information about the dynamics of the considered system.

Analysis of the curves which represent the dynamical behaviour of the system in the plane phase-amplitude gives evidence of very interesting features of dynamics of the system. The shape of these curves depends strongly on the values of parameter α, which is connected with the spring non-linearity. The most intensive energy transfer between the system and its surroundings is governed by the so-called LPT. Important non-linear dynamical transition-type phenomena are detected, monitored and discussed, amongst others. For $\alpha > \alpha_{LPT}$ amplitudes are much greater than for $\alpha < \alpha_{LPT}$. Moreover, it has been shown that the shape of the amplitude modulation curves changes with the value of α.

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