Non-linear Phenomena Exhibited by Flexible Cylindrical and Sector Shells

V.A. Krysko, J. Awrejcewicz, I.V. Papkova, V.B. Baiburin, and T.V. Yakovleva

Abstract Vibrations of flexible cylindrical and sector shells subjected to the action of uniformly distributed static loads are studied. The analyzed problems are solved using two methods: the Bubnov–Galerkin method (BGM) and the finite difference method (FDM). Validity and reliability of the results is verified through a comparison to the results obtained by Andreev et al. (Stability of Shells Under Non-Symmetric Deformation. Nauka, Moscow, 1988) in the case of a nonlinear static problem.

1 Introduction

The variety of loading applied plays a crucial role while estimating the strength of materials in numerous constructions working in high-temperature fields. Proper estimation of the construction strength requires the detailed analysis of the elastic-plasticmaterial behavior, initial deflections, interaction of the construction elements, or interaction of those elements with the surrounding medium. The proper estimation of the construction stability requires development of suitable computational algorithms [12].

V.A. Krysko (✉) • I.V. Papkova • T.V. Yakovleva
Department of Mathematics and Modeling, Saratov State Technical University, Politehnicheskaya 77, 410054 Saratov, Russian Federation
e-mail: tak@san.ru; ikravzova@mail.ru; Yan-tan1987@mail.ru

J. Awrejcewicz
Department of Automation, Biomechanics and Mechatronics, Lodz University of Technology, 1/15 Stefanowski Str., 90-924 Lodz, Poland

Department of Vehicles, Warsaw University of Technology, 84 Narbutta Str., 02-524 Warsaw, Poland
e-mail: jan.awrejcewicz@p.lodz.pl

V.B. Baiburin
Department of Information Security of Automated Systems, Saratov State Technical University, Politehnicheskaya 77, 410054 Saratov, Russian Federation
e-mail: tak@san.ru

J. Awrejcewicz (ed.), *Applied Non-Linear Dynamical Systems*, Springer Proceedings in Mathematics & Statistics 93, DOI 10.1007/978-3-319-08266-0_3

Many researchers have applied the Bubnov–Galerkin methods (BGMs) in the Vlasov form as well as Ritz and FDM methods to solve problems of the stability of beams, plates, and shells subjected to the action of a constant transversal load and taking into account the geometric nonlinearities. The mentioned numerical approaches yield reliable and validated results regarding a wide class of both stationary and nonstationary problems of mathematical physics. In the case of periodic loading, chaotic vibrations of the mentioned structural members may appear [3, 13, 14]. The so far mentioned computational approaches reduce the continuous problems to those of finite degrees of freedom [9, 18].

In order to investigate stability loss one may apply a few different criteria. Since stability loss of an arbitrary deformed object takes place in time, therefore it should be studied using various approaches to dynamics. However, a majority of the stability problems of construction can be studied within static approaches, where the equilibrium states are formulated without the inclusion of inertial forces.

Investigation of the stability loss is carried out using a dynamic criterion. Namely, we define it through a buckling of an equilibrium form. Those loads being responsible for the buckling occurrence are further named the critical loads.

We omit here an overview of the fundamental works dealing with the mentioned problems, but we mention the method developed by Feodosev [10] regarding nonlinear problems of shells, which is rather omitted in English-language literature.

In the latter one being originally named by Feodosev as the variational-step method, the system deformation is considered as a process independent of either fast or slow changes of the external load. For this purpose, time is introduced artificially and equations of motion are derived. Nowadays this method refers to the iteration process of finding solutions to nonlinear algebraic equations, where results obtained in each computational step are improved, finally approaching the desired exact solution of the problem. In this method of relaxation a solution to PDEs is reduced to the Cauchy problem of ODEs.

The proposed algorithm is used to solve a wide class of static and dynamic problems. We show a few possibilities of this approach to solve geometrically nonlinear static and dynamic problems. We consider a mechanical system subjected to the action of the transversal uniformly distributed constant load over the shell surface and we consider the load in the form of the impulse with infinite action. Since the problem of a critical static load plays here a crucial role, we briefly describe the known criteria of stability proposed by numerous researchers.

Already Volmir [19] proposed the following criterion: either a fast increase of deflection corresponding to a small decrease of load appears or an inflexion point of the relation $q(w)$ $\left(\frac{\partial^2 q}{\partial w^2} = 0\right)$ occurs. On the other hand a load, where the increased process of time is responsible for the achievement of the first maximum in the load–time characteristic, is treated as the critical one. Kantor, who solved numerous problems of axially symmetric spherical shells using the Ritz method, proposed the following dynamic criterion responsible for the beam buckling [11]. The buckling occurs if in the shell center its deflection achieves $K \cong 2\overline{f}$, where $\overline{f} = f/h$ and f denotes deflection, whereas h is the shell thickness.

In references [7, 16] different criteria are proposed. Namely, the system transits into a new dynamic state with the corresponding zero velocity. It can be explained in the following manner. In the beginning the inertial forces act against the external load, and after transition through zero they change sign and support the action of the external load. It means that in a certain time instant the beam center velocity achieves zero and then a sudden change of deflection occurs. In reference [8] the time instant is taken as the stability loss criterion, where the displacement of an elastic body changes without a change of the associated accelerations and velocities. In some works the problem of dynamical stability loss is reduced to a quasi-dynamical problem. Owing to this approach, the precritical stress of the middle shell process is analyzed via static approaches. There are also works where a dynamic criterion of the stability loss is matched with the occurrence of plastic deformations of shell structures.

In reference [17], arcs were investigated and their buckling process was characterized by two different mechanisms. In the case of the direct buckling mechanism an unstable construction state was realized via symmetric forms. In the case of indirect buckling, the system lost its stability via nonsymmetric forms. Since the system stability loss via symmetric and nonsymmetric forms is qualitatively different, one may expect two different dynamic criteria of the stability loss.

In this work by a critical load we mean limiting load values or the point of inflexion of the relation $w_{max}(q)$. Further on, we will investigate critical loads acting on axially symmetric spherical and conical shells, on a closed cylindrical shell as well as on a spherical sector shell.

2 Shallow Closed Cylindrical Shells

We study shallow shells, i.e., objects in R^3 with the associated curvilinear coordinates $\overline{x}, \overline{y}, \overline{z}$, introduced in the following manner. In the shell body the middle surface $\overline{z} = 0$ is fixed; axes ox and oy overlap the main shell curvatures, whereas axis oz shows curvature surface origin (Fig. 1). In the given coordinates the shell is defined as follows: $\Omega = \left\{\overline{x}, \overline{y}, \overline{z}/(\overline{x}, \overline{y}, \overline{z}) \in [0, \overline{a}] \times [0, \overline{b}] \times \left[-\overline{h}/2, \overline{h}/2\right]\right\}$, where dimensional quantities are denoted by bars.

The governing nonlinear dynamics of the shell shown in Fig. 1 is obtained assuming that the shell material is isotropic, homogeneous, and elastic and it satisfies the Kirchhoff–Love hypotheses. Furthermore, we assume that the length of the shell fiber along shell thickness remains unchanged [19].

Therefore, in the nondimensional form, the equation of motion of the shell element as well as the deformation compatibility equations have the following nondimensional forms:

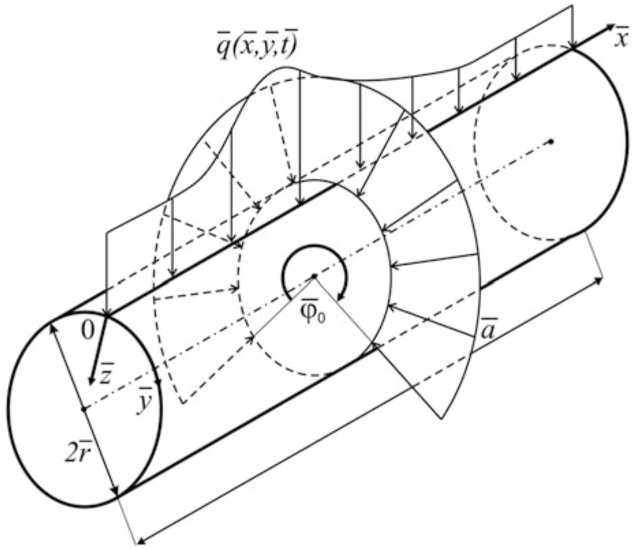

Fig. 1 Computational scheme of a cylindrical shell

$$\left[\frac{1}{\lambda^2}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial x^2}+\lambda^2\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial y^2}+2(1-\mu)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}+\mu\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial y^2}+\right.\right.$$

$$\left.\left.+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial x^2}\right)\right]-\nabla_k^2 F-L(w,F)+Mq(t)-\left(\frac{\partial^2 w}{\partial t^2}+\varepsilon\frac{\partial w}{\partial t}\right)=0,$$

$$\left[\left(\lambda^2\frac{\partial^2 F}{\partial y^2}-\mu\frac{\partial^2 F}{\partial x^2}\right)\frac{\partial^2(\cdot)}{\partial y^2}+\left(\frac{1}{\lambda^2}\frac{\partial^2 F}{\partial x^2}-\mu\frac{\partial^2 F}{\partial y^2}\right)\frac{\partial^2(\cdot)}{\partial x^2}+\right. \quad (1)$$

$$\left.+2(1+\mu)\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right]+\nabla_k^2 w+\frac{1}{2}L(w,w)=0.$$

The following relations hold between dimensional and nondimensional quantities:

$$w=h\overline{w},\quad F=Eh^2\overline{F},\quad t=t_0\overline{t},\quad \varepsilon=\overline{\varepsilon}/\tau,\quad x=L\overline{x},\quad y=R\overline{y},$$

$$k_y=\overline{k}_y\frac{h}{R^2}\quad (k_x=0),\quad q=\overline{q}\frac{Eh^4}{L^2R^2},\quad \tau=\frac{LR}{h}\sqrt{\frac{\rho}{Eg}}, \quad (2)$$

$$M=k_y^2,\quad \lambda=\frac{L}{R},$$

where L and $R = R_y$ correspond to the shell length and radius, respectively. In addition, we have t, time; ε, damping coefficient; $\mu = 0.3$; and $q(x, y, t)$, transversal load. One of the following boundary conditions is taken:

1. Moving clamping

$$w = 0; \quad \frac{\partial w}{\partial x} = 0; \quad F = 0; \quad \frac{\partial F}{\partial x} = 0 \quad \text{for} \quad x = 0; 1, \tag{3}$$

$$w = g(x, y, t); \ \frac{\partial w}{\partial y} = p(x, y, t); \ F = u(x, y, t); \ \frac{\partial F}{\partial y} = v(x, y, t) \text{ for } y = 0; \xi.$$

2. Pinned support

$$w = 0; \ \frac{\partial w}{\partial x} = 0; \ F = 0; \ \frac{\partial^2 F}{\partial x^2} = 0 \text{ for } x = 0; 1, \tag{4}$$

$$w = g(x, y, t); \ \frac{\partial w}{\partial y} = p(x, y, t); \ F = u(x, y, t); \ \frac{\partial F}{\partial y} = v(x, y, t) \text{ for } y = 0; \xi.$$

3. Moving clamping with ribs

$$w = 0; \quad \frac{\partial^2 w}{\partial x^2} = 0; \quad F = 0; \quad \frac{\partial F}{\partial x} = 0 \quad \text{for} \quad x = 0; 1, \tag{5}$$

$$w = g(x, y, t); \ \frac{\partial w}{\partial y} = p(x, y, t); \ F = u(x, y, t); \ \frac{\partial F}{\partial y} = v(x, y, t) \text{ for } y = 0; \xi.$$

4. Pinned support with flexible ribs

$$w = 0; \quad \frac{\partial^2 w}{\partial x^2} = 0; \quad F = 0; \quad \frac{\partial^2 F}{\partial x^2} = 0 \quad \text{for} \quad x = 0; 1, \tag{6}$$

$$w = g(x, y, t); \ \frac{\partial^2 w}{\partial y^2} = r(x, y, t); \ F = u(x, y, t); \ \frac{\partial^2 F}{\partial y^2} = z(x, y, t) \text{ for } y = 0; \xi.$$

Here we take $\xi = 2\pi$ for a closed cylindrical shell. In addition, the following initial conditions are applied:

$$w|_{t=0} = w_0, \qquad \dot{w}|_{t=0} = w_0. \tag{7}$$

2.1 The Bubnov–Galerkin Method (BGM)

After application of the BGM the following system of algebraic-differential equations is obtained:

$$\begin{gathered}\mathbf{G}(\ddot{\mathbf{A}}+\boldsymbol{\varepsilon}\dot{\mathbf{A}})+\mathbf{HA}+\mathbf{C}_1\mathbf{B}+\mathbf{D}_1\mathbf{AB}=\mathbf{Q}q(t),\\ \mathbf{C}_2\mathbf{A}+\mathbf{PB}+\mathbf{D}_2\mathbf{AA}=0,\end{gathered} \tag{8}$$

where $\mathbf{H}=\|H_{ijrs}\|$, $\mathbf{G}=\|G_{ijrs}\|$, $\mathbf{C}_1=\|C_{1ijrs}\|$, $\mathbf{C}_2=\|C_{2ijrs}\|$, $\mathbf{D}_1=\|D_{1ijklrs}\|$, $\mathbf{D}_2=\|D_{2ijklrs}\|$, $\mathbf{P}=\|P_{ijrs}\|$ - square matrices of dimensions $2\cdot N_1\cdot N_2\times 2\cdot N_1\cdot N_2$, $\mathbf{A}=\|A_{ij}\|$, $\mathbf{B}=\|B_{ij}\|$, $\mathbf{Q}=\|Q_{ij}\|$ matrices of dimension $2\cdot N_1\cdot N_2\times 1$.

The second equation of system (8) is solved regarding **B** on each of the computational steps:

$$\mathbf{B}=[-\mathbf{P}^{-1}\mathbf{D}_2\mathbf{A}-\mathbf{P}^{-1}\mathbf{C}_2]\mathbf{A}. \tag{9}$$

Multiplying by $\mathbf{G}^{-1}$ and taking $\dot{\mathbf{A}}=\mathbf{R}$, the following Cauchy problem is formulated for nonlinear ODEs:

$$\begin{aligned}&\dot{\mathbf{R}}=-\bar{\varepsilon}\mathbf{R}-[\mathbf{G}^{-1}\mathbf{C}_1+\mathbf{G}^{-1}\mathbf{D}_1\mathbf{A}]\cdot\mathbf{B}-\mathbf{G}^{-1}\mathbf{HA}+\mathbf{G}^{-1}\mathbf{Q}q(\bar{t}),\\ &\dot{\mathbf{A}}=\mathbf{R}.\end{aligned} \tag{10}$$

It is solved via the fourth-order Runge–Kutta method, and the computational step in time is chosen using the Runge rule. We apply the method of relaxation for the closed cylindrical shells with $\lambda=2$ and we compare our results with the solution obtained by Andreev et al. [1] for the corresponding static problem. We consider the case of transversal external load whose location is defined by the central angle φ_0. In order to get $q_{cr}(\varphi_0)$ we need to construct a set $\{q_i, w_i\}$ for $\forall\varphi_0\in[0;2\pi]$, which yields the critical load q_{cr}. As it has been shown in [15], an increase in the number of approximations yields a remarkable improvement of the obtained results.

The following conclusion can be formulated: in the case of nonhomogeneous load, the use of a small number of the series terms yields large computational errors and the obtained results depend essentially on the number of introduced approximations. However, the situation changes qualitatively beginning from $N=13$. Namely, the dynamical properties of the cylindrical shell are stabilized and a further increase of N does not improve the obtained results either qualitatively or quantitatively. Therefore, beginning from $N=13$, a convergent series is obtained and all further computations are carried out for $N=13$. Consequently, we constructed the relation of the critical loads versus width of the pressure zone $\bar{q}_{cr}(\varphi_0)$ for $N=13$ reported in [15].

Dependencies $\bar{q}_{cr}(\varphi_0)$ reported by Andreev et al. [1] for the closed cylindrical shell for $\lambda=2$ are in agreement with the results obtained by our method. Hence, the obtained results indicate high efficiency of the proposed method for solving static problems.

3 Shallow Sector Shells

Consider now a non-axially symmetric spherical shell in R^2 in the polar coordinates bounded by a contour Γ, introduced in the following way: $\overline{\Omega} = \Omega + \Gamma = \{(r, \theta, z)|r \in [0, b], \theta \in [0, \theta_k], z \in [-h/2, h/2]$. Equations governing the dynamics of shallow shells are obtained from a system of equations of the rectangular spherical shell via transition to the polar coordinates:

$$\begin{gathered} w'' + \varepsilon w' = -\nabla^2\nabla^2 w + N(w, F) + \nabla^2 F + 4q, \\ \nabla^2\nabla^2 F = -\nabla^2 w - N(w, w), \end{gathered} \tag{11}$$

where

$$\nabla^2(\cdot) = \frac{\partial^2(\cdot)}{\partial r^2} + \frac{1}{r}\frac{\partial(\cdot)}{\partial r} + \frac{1}{r^2}\frac{\partial^2(\cdot)}{\partial \theta^2},$$

$$\nabla^2\nabla^2(\cdot) = \frac{\partial^4(\cdot)}{\partial r^4} + \frac{2}{r}\frac{\partial^3(\cdot)}{\partial r^3} - \frac{1}{r^2}\frac{\partial(\cdot)}{\partial r^2} + \frac{1}{r^3}\frac{\partial(\cdot)}{\partial r} +$$

$$+\frac{2}{r^2}\frac{\partial^4(\cdot)}{\partial \theta^2 \partial r^2} - \frac{2}{r^3}\frac{\partial^3(\cdot)}{\partial \theta^2 \partial r} + \frac{4}{r^4}\frac{\partial^2(\cdot)}{\partial \theta^2} + \frac{1}{r^4}\frac{\partial^4(\cdot)}{\partial \theta^4},$$

$$N(w, F) = \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right) + \frac{\partial^2 F}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) - \tag{12}$$

$$-2 \cdot \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right),$$

$$N(w, w) = 2 \cdot \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) - 2 \cdot \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right]^2.$$

Boundary conditions follow:

1. Pinned support of arc slices

$$w = 0, \quad \frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r} = 0, \quad F = 0, \quad \frac{\partial F}{\partial r} = 0. \tag{13}$$

2. Pinned support of radial slices

$$w = 0, \quad \frac{\partial^2 w}{\partial \theta^2} = 0, \quad F = 0, \quad \frac{\partial^2 F}{\partial \theta^2} = 0. \tag{14}$$

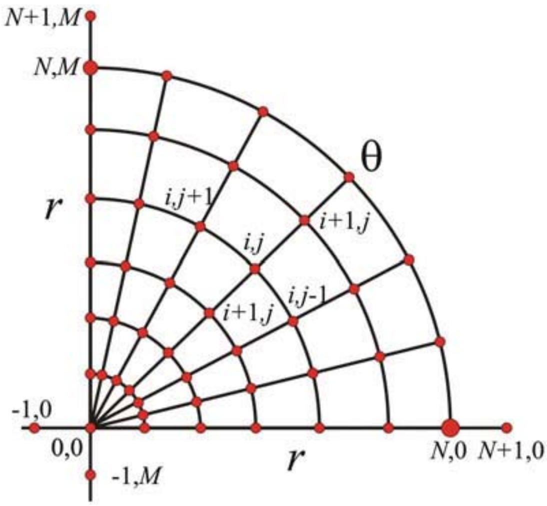

Fig. 2 Mesh of a sector shell

3. Sliding clamping of arc slices

$$w = 0, \quad \frac{\partial w}{\partial r} = 0, \quad F = 0, \quad \frac{\partial F}{\partial r} = 0. \tag{15}$$

4. Sliding clamping of radial slices

$$w = 0, \quad \frac{\partial w}{\partial \theta} = 0, \quad F = 0, \quad \frac{\partial^2 F}{\partial \theta^2} = 0. \tag{16}$$

Initial conditions are as follows:

$$w = f_1(r,\theta) = 0, \quad w' = f_2(r,\theta) = 0 \quad \text{for} \quad t = 0. \tag{17}$$

3.1 Finite Difference Method

In order to reduce the continuous system governed by (12)–(17) to a lumped system by the FDM with the approximation $O(\Delta^2)$ versus spatial coordinates r and θ (Fig. 2), the following difference operators are applied:

$$-\Lambda(\Lambda w) + \Lambda_{rr} w(\Lambda F + \Lambda_{rr} F) + \Lambda_{rr} F(\Lambda w + \Lambda_{rr} w) -$$
$$-2 \cdot \Lambda_{r\theta} w \Lambda_{r\theta} F + \Lambda F + 4q_i = (w_{tt} + \varepsilon w_t)_{i,j}, \tag{18}$$
$$\Lambda(\Lambda F) = -\Lambda_{rr} w(\Lambda w + \Lambda_{rr} w) + (\Lambda_{r\theta} w)^2 - \Lambda w,$$

where

$$\Lambda(\cdot) = \Lambda_{rr}(\cdot) + \Lambda_r(\cdot),\ \Lambda_r(\cdot) = \frac{1}{r_i^2}(\cdot)_r,\ \Lambda_{rr}(\cdot) = (\cdot)_{rr},\ \Lambda_{r\theta}(\cdot) = -\frac{1}{r_i^2}(\cdot)_\theta + \frac{1}{r_i}(\cdot)_{r\theta},$$

$$\Lambda_{rr}(\cdot) = \frac{1}{\Delta^2}[(\cdot)_{i+1} - 2(\cdot)_i + (\cdot)_{i-1}], \quad \Lambda_r(\cdot) = \frac{1}{2\cdot\Delta\cdot r_i^2}[(\cdot)_{i+1} - (\cdot)_{i-1}].$$

Boundary conditions:

1. Pinned support of arc slices

$$w_{N,j} = 0, \quad \Lambda_{rr}w - \frac{\nu}{b}\Lambda_r w = 0, \quad F_{N,j} = 0, \quad \Lambda_r w = 0, \quad j = 1,\ldots,M-1. \tag{19}$$

2. Pinned support of radial slices

$$w_{i,j} = 0, \quad \Lambda_{\theta\theta}w = 0, \quad F_{i,j} = 0, \quad \Lambda_{\theta\theta}F = 0, \quad j = 0, M, \quad i = 0,\ldots,N. \tag{20}$$

3. Sliding clamping of arc slices

$$w_{N,j} = 0, \quad \Lambda_r w = 0, \quad F_{N,j} = 0, \quad \Lambda_r F = 0, \quad j = 1,\ldots,M-1. \tag{21}$$

4. Sliding clamping of radial slices

$$w_{i,j} = 0, \quad \Lambda_{\theta\theta}w = 0, \quad F_{i,j} = 0, \quad \Lambda_{\theta\theta}F = 0, \quad j = 0, M, \quad i = 0,\ldots,N. \tag{22}$$

The system of (18)–(22) should be supplemented by conditions to be satisfied in the shell cusp and the matching conditions. In the majority of cases it is assumed that a shell has a circular hole of small dimension in its cusp, and this assumption does not influence computational results essentially. In this work, while solving nonsymmetric problems for $\theta = 2\cdot\pi$, the approximating functions in the point $r = 0$ are interpolated by the Lagrange formula of the second order. We have

$$f_{0,j} = 3\cdot f_{1,j} - 3\cdot f_{2,j} + f_{3,j}, \tag{23}$$

where $f_{i,j} = f(r_i)_j$, $r_i = i\cdot h$ $(i = 0,1,2,3)$, $0 \le j \le M-1$, and h is the distance between the nodes of interpolation. In the case of a point lying out of the contour the following symmetry condition holds:

$$f_{-1,j} = f_{1,j} \quad \text{for} \quad 0 \le j \le M-1. \tag{24}$$

Matching conditions for non-axially symmetric problems $\theta = 2\cdot\pi$ follow:

$$w_{i,j} = w_{i,M+j}, \quad F_{i,j} = F_{i,M+j} \quad \text{for} \quad j = 0; -1, \quad 0 \le i \le N-1. \tag{25}$$

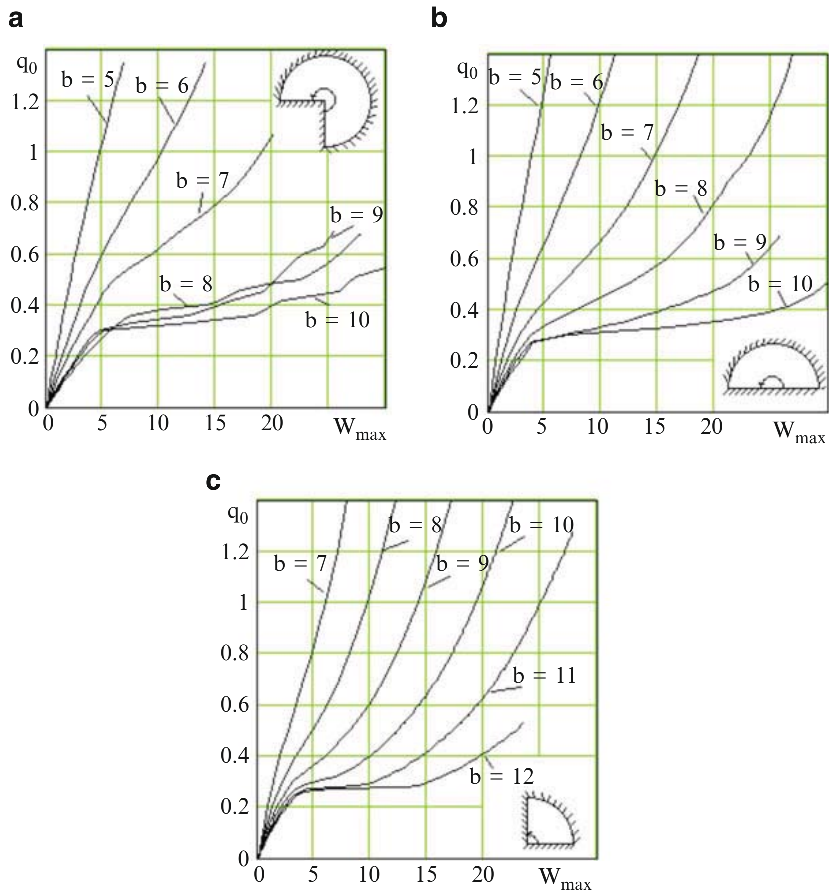

Fig. 3 Computational results of sector shells analysis

The Cauchy problem (18)–(25) is solved via the fourth-order Runge–Kutta method, where the computational step has been chosen due to the Runge rule [2, 4–6].

Figure 3 shows the dependencies $q(w)$ for sector angles $\theta_k = \frac{3}{2} \cdot \pi$ (a), for $\theta_k = \pi$ (b), and for $\theta_k = \frac{\pi}{2}$ (c) and for various shallow parameters for $\theta_k = \frac{3}{2} \cdot \pi$ and $\theta_k = \pi$ ($b = 5, 6, 7, 8, 9, 10$), and for $\theta_k = \frac{\pi}{2}$ for the shallow parameter $b = 7, 8, 9, 10, 11, 12$ (lower values of b are not used, since the sector shell behaves like a plate).

The obtained graphs imply the following conclusions. Beginning from a certain value of the parameter b the critical points appear. For a shell with $\theta_k = \frac{3}{2} \cdot \pi; \pi; \frac{\pi}{2}$ and $b = 8, 9$ and 11, respectively, the jump-type buckling occurs. Figure 4 illustrates the curves of equal deflections (isoclines) for all studied angles $\theta = \frac{\pi}{2}; \pi; \frac{3 \cdot \pi}{2}$. In what follows we compare the behavior of curves for the critical and post-critical load q_0 for different θ and b. For $\theta = \frac{\pi}{2}$ and $\theta = \pi$ and for an arbitrary shallow parameter the obtained results coincide and the maximum deflection is achieved on the intersection of the bisectrix of angle θ and the central shell radius. For $\theta = \frac{3 \cdot \pi}{2}$ and for $b = 7$ the jump-type buckling is not observed, and the system dynamics

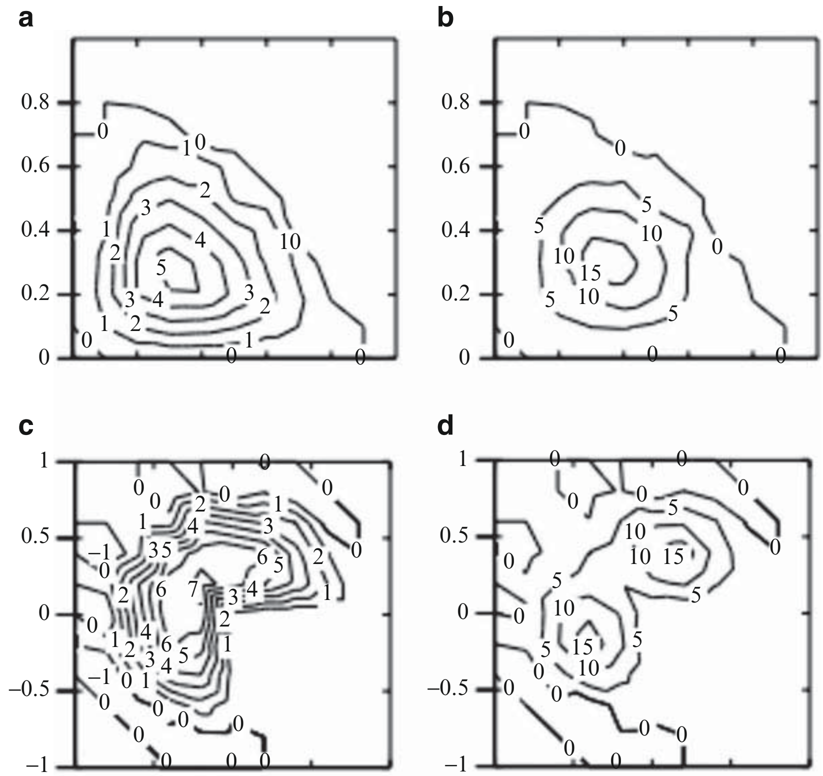

Fig. 4 Computational results of sector shells (isoclines) for $\theta = \frac{\pi}{2}, b = 12, q_0 = 0.245$ (**a**), $\theta = \frac{\pi}{2}, b = 12, q_0 = 0.246$ (**b**), $\theta = \frac{3 \cdot \pi}{2}, b = 10, q_0 = 0.27$ (**c**) and $\theta = \frac{3 \cdot \pi}{2}, b = 10, q_0 = 0.28$ (**d**)

is not changed qualitatively. In Fig. 3a the same sector angle and the first-order discontinuity imply qualitative changes in the location of isoclines for both critical and post-critical load (see Fig. 4). For $\theta = \frac{\pi}{2}$, $b = 12$ a fast shell deflection is observed but without a change of the shell form. For $\theta = \frac{3 \cdot \pi}{2}$, $b = 10$ for the critical load, the maximum shell deflection occurs in the middle of the bisectrix, whereas in the case of post-critical load two zones of maximum deflection appear, being symmetric with respect to the angle bisectrix.

4 Concluding Remarks

In this work the relaxation method has been applied to study flexible cylindrical and sector shells. The Bubnov–Galerkin and the finite difference approaches allowed us to reduce the problems to the algebraic-ordinary differential equations. A few interesting nonlinear phenomena exhibited by the analyzed continuous systems have been reported.

Acknowledgements This work has been supported by the grant RFBR No. 12-01-31204 and the National Science Centre of Poland under the grant MAESTRO 2, No. 2012/04/A/ST8/00738, for the years 2013–2016.

References

1. Andreev, L.V., Obodan, N.I., Lebedev, A.G.: Stability of Shells Under Non-Symmetric Deformation. Nauka, Moscow (1988) (in Russian)
2. Awrejcewicz, J., Krysko, V.A., Krysko, A.V.: Thermodynamics of Plates and Shells. Springer, Berlin (2007)
3. Awrejcewicz, J., Krylova, E.Yu., Papkova, I.V., Yakovleva, T.V., Krysko, V.A.: On the memory of nonlinear differential equations in theory of plates. Vestnik Niznegorodskogo Universiteta im. Lobachevskogo **4**(2), 21–23 (2011) (in Russian)
4. Awrejcewicz, J., Krysko, A.V., Papkova, I.V., Krysko, V.A.: Routes to chaos in continuous mechanical systems. Part 1: mathematical models and solution methods. Chaos Solitons Fractals **45**, 687–708 (2012)
5. Awrejcewicz, J., Krysko, A.V., Papkova, I.V., Krysko, V.A.: Routes to chaos in continuous mechanical systems: Part 2. Modelling transitions from regular to chaotic dynamics. Chaos Solitons Fractals **45**, 709–720 (2012)
6. Awrejcewicz, J., Krysko, A.V., Papkova, I.V., Krysko, V.A.: Routes to chaos in continuous mechanical systems. Part 3: the Lyapunov exponents, hyper, hyper-hyper and spatial-temporal chaos. Chaos, Solitons Fractals **45**, 721–736 (2012)
7. Bakhtieva, L.U.: Investigation of stability of thin shells and plates subjected to dynamic loads. Ph.D. Thesis, Kazan (1981) (in Russian)
8. Darevskiy, V.M.: Nonlinear equations in theory of shells and their linearizations in problems of stability. In: Proceedings of VI Russian Federation Conference on Theory of Plates and Shells, pp. 355–368. Nauka, Moscow (1969) (in Russian)
9. Deresiewicz, H.: Solution of the equations of thermoelasticity. In: Proceedings of 3rd National Congress of Applied Mechanics ASME, pp. 685–690, 1958
10. Feodosev, V.I.: On the solution of non-linear stability problems of deformable systems. PMM **27**(2), 265–274 (1963) (in Russian)
11. Kantor, B.Ya.: On the Non-linear Theory of Shells. Dynamics and Strength of Machines. Kharkiv National University, Kharkiv (1967)
12. Krysko, V.A., Awrejcewicz, J., Kuznetsova, E.S.: Chaotic vibrations of shells in a temperature field. In: Proceedings of the International Conference on Engineering Dynamics, Carvoeiro, Algarve, Portugal, , pp. 21–28, 2007
13. Krysko, V.A., Saltykova, O.A., Yakovleva, T.V.: Nonlinear dynamics of antennas in cosmonautic coupling tools. Izvestia VUZ. Aviation Tech. **2**, 60–62 (2011) (in Russian)
14. Krysko, A.V., Koch, M.I., Yakovleva, T.V., Nackenhorst, U., Krysko, V.A.: Chaotic nonlinear dynamics of cantilever beams under the action of signs-variables loads. In: PAMM, Special Issue: 82nd Annual Meeting of the International Association of Applied Mathematics and Mechanics (GAMM), vol. 11(1), pp. 327–328, Graz, 2011
15. Krysko, V.A., Awrejcewicz, J., Papkova, I.V., Baiburin, V.B., Yakovleva, T.V.: Method of relaxation in theory of flexible shells. In: Dynamical Systems Theory. pp. 477–490. TU of Lodz Press, Lodz (2013)
16. Kuznetsov, E.B.: On action of dynamic load on systems with buckling. Probl. Appl. Mech. 12–35 (1974) (in Russian)

17. Lock, M.H.: Snapping of a shallow sinusoidal arch under a step pressure load. AIAA J. **4**(7), 31–41 (1966)
18. Takezono, S., Tao, K., Inamura, E., Inoue, M.: Thermal stress and deformation in functionally graded material shells of revolution under thermal loading due to fluid. JSME Int. J. Ser. A Solid Mech. Mater. Eng. **39**(4), 573–581 (1996)
19. Volmir, A.S.: Stability of Elastic Systems. Fizmatgiz, Moscow (1963) (in Russian)