Mathematical model of a single-dimension multi-parameter oscillator based on a three-phase core-less linear motor

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Abstract: This paper uses the example of a three-phase core-less linear motor to create a mathematical model of single-dimension multi-parameter oscillator. The oscillator is composed of an U-shaped stationary guide-way with permanent magnets corresponding to the motor's stator, and a movable set of coils subjected to the alternating electrical voltage (equivalent of motor's forcer). The system's parameters are either mechanical (number of magnets and coils, size of the magnets, distances between magnets, size and shape of the coils) or electromagnetic (auxiliary magnetic field and permeability). A voltage applied to each of the coils serves as the external excitation while displacement of the coils is the output parameter. Faraday's law of induction and Lorentz force yield base laws for creating the model and a Gaussian function of distance is used to determine the value of auxiliary magnetic field. The solution to the introduced mathematical model of the studied mechatronic system (consisting of the partial differential and integral equations) have been compared with the experimental results showing a good coincidence.

1. Introduction

The idea of creating the model, presented in this paper, arose during the making of a simple lab stand. Its purpose was to demonstrate the advantages of modern linear motors and drives over the traditional solutions such as feed screw conveyors. During the first test starts however, the stand's motor movement was jerky, erratic and sometimes unpredictable. It was later revealed that those errors were merely a result of faulty assembly and were quickly subdued. Initially however it was believed that a comprehensive mathematical model was needed to fix the issues. The model was created and the result yielded were more interesting than anticipated.

2. Lab stand (validation platform)

The lab stand used as a base for constructing the model consists of HIWIN's coreless linear motor and Copley-Controls' servo-drive. The forcer (inductor) is capable of moving at speeds of up to 5 meters per second with a load of 45N. An analogue optical linear encoder can read the forcer's position with a resolution of 0,1 µm and the entire system's positioning error not trailing far behind. Three U-shaped stators with permanent magnets were used to create the motors magnetic guide-way giving the stand a theoretical maximum stroke of 830 mm and an actual stroke of 780 mm. Two high quality linear guide-way provide for a swift and quiet motion. The stand program allows it to work with both manual input and automatically based on a signal from external devices. [4]

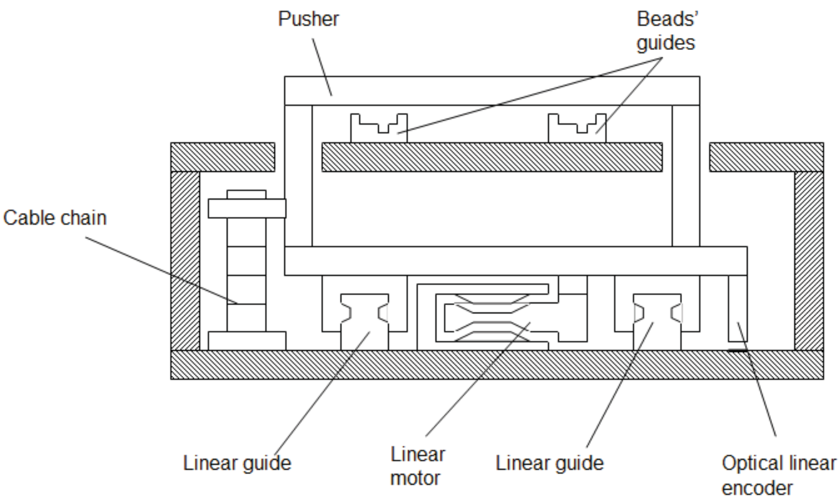

Figure 1. Lab stands diagram

As the most interesting scientifically only the motor will be modeled in this paper. The stand will serve merely as a validation platform.

3. Model construction

A single motor winding can be modeled as a perfectly round conductor loop, with a certain voltage function U_g applied to it. The means and exact spot of this application is omitted as unimportant for the workings of the model. Let this loop be placed in the vicinity of a C-shaped magnet in a way depicted in Figure 2.

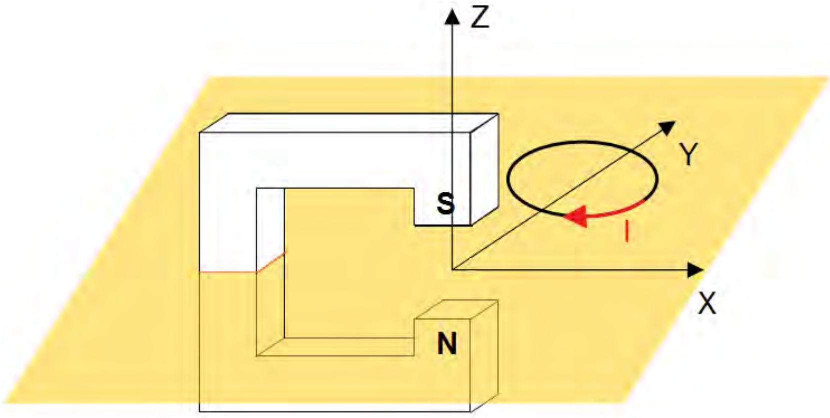

Figure 2. Model of a single winding

Assuming that the loop cannot deform and can only move along the direction of y-axis it shall always remain a circle with a center placed on y-axis. Every infinitely small slice of the loop contained between φ and φ+dφ is subjected to Lorentz force working perpendicularly to its tangent (see Figure 3).

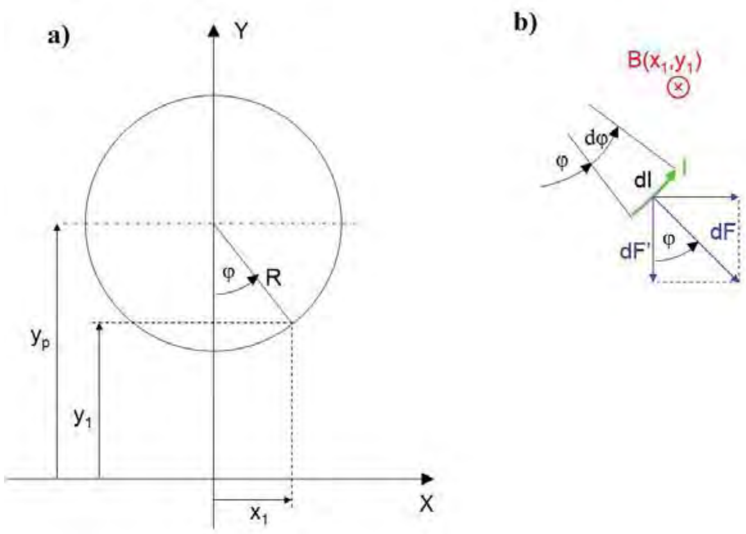

Figure 3. Single loop overview: (a) - coordinate system; (b) - Lorentz force of the elemental slice

The value of that force equals

$$d\vec{F} = \vec{B} \times \vec{i}, \tag{1}$$

$$d\vec{F} = I\vec{B} \times d\vec{l}, \tag{2}$$

where i and B are the vector of electric current and magnetic induction respectively. In every point of a conductor loop the i vector's direction is tangent to the coil while it's value can be treated as constant equal to I.

Since the model cannot move in any direction other than along y-axis the only significant component of the Lorentz force is one parallel to that axis. It shall be defined as

$$dF' = dF\cos\varphi = B' I\, dl \cos\varphi, \tag{3}$$

After integration

$$F = \oint_L B' I \cos\varphi\, dl, \tag{4}$$

In this equation F is the y-component of total force acting on the coil and B' is the z-component of magnetic field. As the loop is stationary along z-axis and assuming it is placed on a plane located ideally between the magnet poles, the distribution of B' can be approximated using Gaussian function in the following way

$$B' = B(x, y, z) = B(x, y) = B_0 \exp\left[-\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)\right], \tag{5}$$

where B_0 is magnetic induction constant dependent of the material used for the magnet and permeability of the environment. Symbols σ_x and σ_y represent magnet poles dimension in their respective axes, Greek letters are used to keep consistency with Gaussian function nomenclature. [2]

Each point on and inside the loop can be represented in Cartesian as well as polar coordinate system using the following transformation

$$x = r\sin\varphi, \quad y = y_p + r\cos\varphi, \tag{6}$$

where r is the distance of the point from the center of the loop and y_p is the y-coordinate of the center in Cartesian system. After converting Eq. (5) to polar system and the line integral to ordinary integral, the total force acting on the coil equals

$$F = I\int_0^{2\pi} B_0 \exp\left[-\frac{(R\sin\varphi)^2}{\sigma_x^2} - \frac{(y_p + R\cos\varphi)^2}{\sigma_y^2}\right] R\cos\varphi\, d\varphi, \tag{7}$$

The value of current flowing through the coil is proportional to the sum of voltage function U_g and motion induced voltage of U_i and inversely proportional to electrical resistance R_E. It can be calculated using

$$I = \frac{U_g + U_i}{R_E}. \tag{8}$$

The induced voltage can be calculated using Faraday's law of induction which states that it is equal to the change in magnetic flux, so we get

$$U_i = -\frac{d\Phi}{dt} = -\dot{\Phi}. \tag{9}$$

The flux flowing through a closed loop is equal to

$$\Phi = \iint_S B(x, y)dS, \tag{10}$$

where S is the area inside of the loop. After inclusion of Eq. (5) and conversion to polar coordinate system the induced voltage can be presented in the following ordinary integral form

$$U_i = \frac{d}{dt}\int_0^R \int_0^{2\pi} B_0 \, r \exp\left[-\frac{(r\sin\varphi)^2}{\sigma_x^2} - \frac{(y_p + r\cos\varphi)^2}{\sigma_y^2} \right] d\varphi \, dr. \tag{11}$$

Equations (11) and (7) are sufficient to build a functional model of a single loop in the vicinity of a single magnet. The inductor of a functional motor can be modeled with three pairs of those equation with y_p differing over a time-independent value (representing the displacement of the coils).

The guide-way can be represented by an infinite amount of C-shape magnets evenly distributed along the x-axis with poles of every two neighboring ones placed oppositely (see Figure 4).

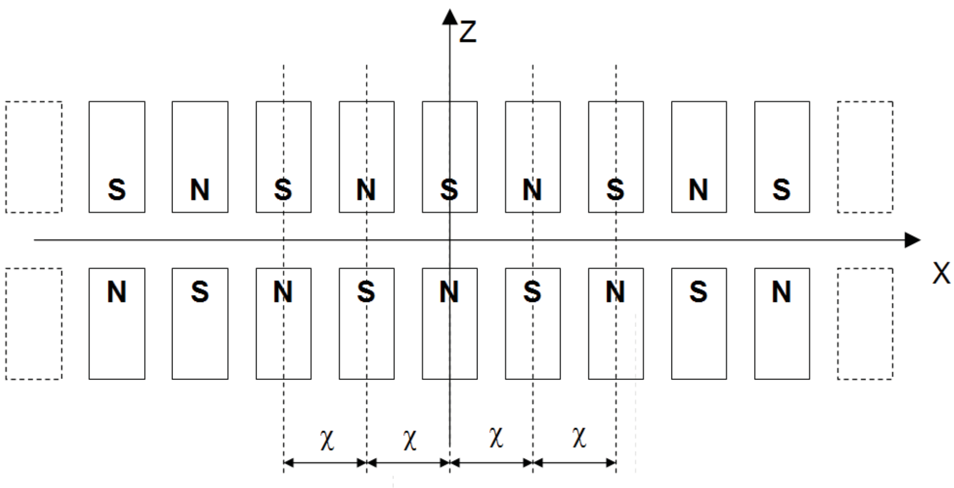

Figure 4. Infinitely long magnetic guide-way model

For such a guide-way the distribution function of magnetic field takes form of an infinite series and can be estimated by the following formula

$$B(x,y) = B_0 \exp\left[-\left(\frac{x^2}{\sigma_x^2}\right)\right]\sum_{i=0}^{\infty}(-1)^i \left[\exp\left(-\frac{(y+i\chi)^2}{\sigma_y^2}\right)+\exp\left(-\frac{(y-i\chi)^2}{\sigma_y^2}\right)\right]. \quad (12)$$

This equation is however rather inelegant and cumbersome. Let there be a correlation between the magnets size (in y-axis) and distribution, expressed numerically by a factor k such that

$$k = \frac{\chi}{\sigma_y}. \quad (13)$$

It can be easily proven [1] (see Figure 5) that there exists such a k that

$$B_1(x,y) = B_0 \exp\left[-\left(\frac{x^2}{\sigma_x^2}\right)\right] \cos\left(\frac{y}{\chi}2\pi\right) \approx B(x,y). \quad (14)$$

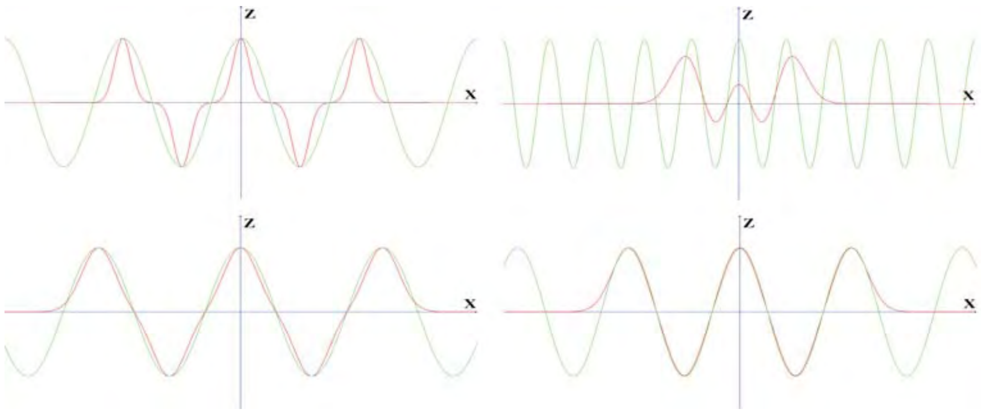

Figure 5. Comparison of distributions given by Eq. (12) - red line and (14) - green line for three different k-factors

For j-coil let y_j, F_j, U_j, U_{gj} and U_{ij} be the values of: the placement of the center, force applied to and total, external and induced voltage on that coil, respectively. Assuming that the coils are evenly displaced

$$y_i = y_0 + a(1-i), \quad (15)$$

where y_0 is the position of inductor as a whole.

The resulted oscillator can be fully described with the following set of equations

$$\ddot{y}_i = \frac{1}{m}\sum_{j=1}^{3} F_j, \tag{16}$$

$$F_j = B_0\, R \frac{U_{gj} + U_{ij}}{R_E} \int_0^{2\pi} \exp\left[-\frac{(R\sin\varphi)^2}{\sigma_x^2}\right] \cos\left(\frac{R\cos\varphi + y_j}{\chi}\right) \cos\varphi\, d\varphi, \tag{17}$$

$$U_{ij} = \frac{d}{dt}\int_0^R \int_0^{2\pi} B_0\, r \exp\left[-\frac{(r\sin\varphi)^2}{\sigma_x^2}\right] \cos\left(\frac{r\cos\varphi + y_j}{\chi}\right) \cdot \varphi \cdot r. \tag{18}$$

Hence the oscillator model is complete

4. Validation

Based on the Equations (16)-(18) a computer simulation was created. Due to their complexity and presence of analytically unsolvable integrals standard mathematical environments' simulators such as scilab's xcos were insufficient. [3] Instead a standalone application, with aforementioned equations embedded in its code, was written in Pascal. It's simple graphical interface allows changing each and every of model's parameter's such as: size of coils, their distribution, electrical resistance etc. It also incorporates an option to shape the function of external excitation (voltage on each coil) to almost any imaginable form and magnitude.

Accordingly the stand's servo-drive can be set in "maintenance mode" allowing for numerous options. The most desirable is the possibility of direct control from a PC desktop. The tuning function allows for, amongst other things, subjecting the coils to a given voltage function. The variety of those functions is scarce but sufficient, all of them have the form

$$U_{gj} = U_M \sin\left(\omega t + (j-1)\frac{2}{3}\pi\right), \tag{19}$$

with U_M (maximum voltage) and ω (angular frequency) changeable in time along a step, a sawtooth or a sinusoidal function. For the purpose of validation a step function of maximum voltage was used.

The computer model was supplied with the following set of parameters, taken from the motor's documentation as well as from direct measurement, so that it resembles the actual motor as closely as possible.

Table 1 - Parameters used for the model

Parameter	Symbol	Value
Coil radius	R	40 [mm]
Coil displacement	a	12 [mm]
Electrical resistance	R_E	6,7 [Ω]
Magnets' field strength	B_0	1,5 [T]
Magnets' width (x-axis)	σ_x	47 [mm]
Magnets' length (y-axis)	σ_y	13 [mm]
Magnets' displacement	χ	32 [mm]
Inductor's mass	M	0,31 [kg]
Maximum voltage	U_M	1,5 [V]
Voltage frequency	ω	20 [rad/s]
Voltage step function frequency	f	1 [Hz]

Figure 6 displays the function of maximum voltage and the resulting voltage on one of the coils taken from the programmed simulation.

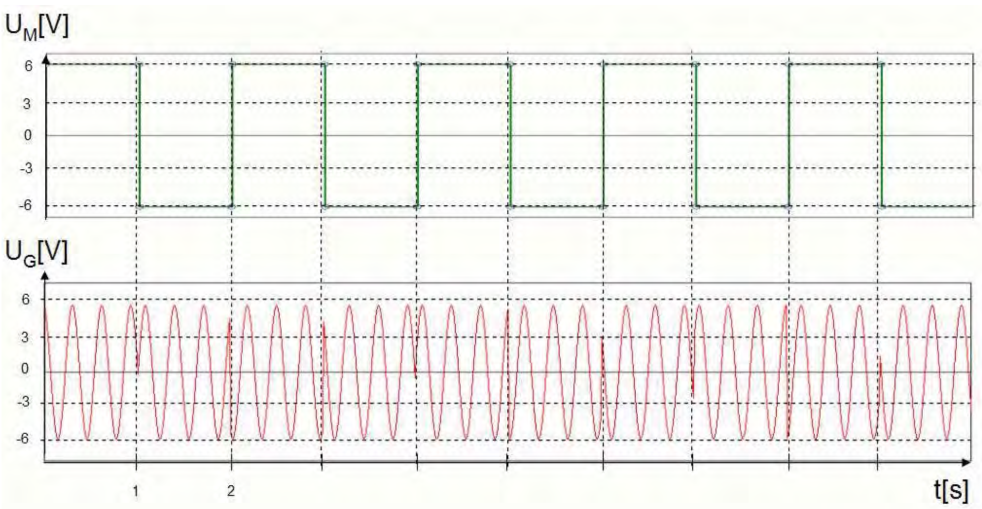

Figure 6. Function of maximum voltage (top most graph) and voltage on the first coil (bottom most graph)

Exactly the same function of voltage was applied to actual motor coils. The result of both the experiment and simulation are presented below.

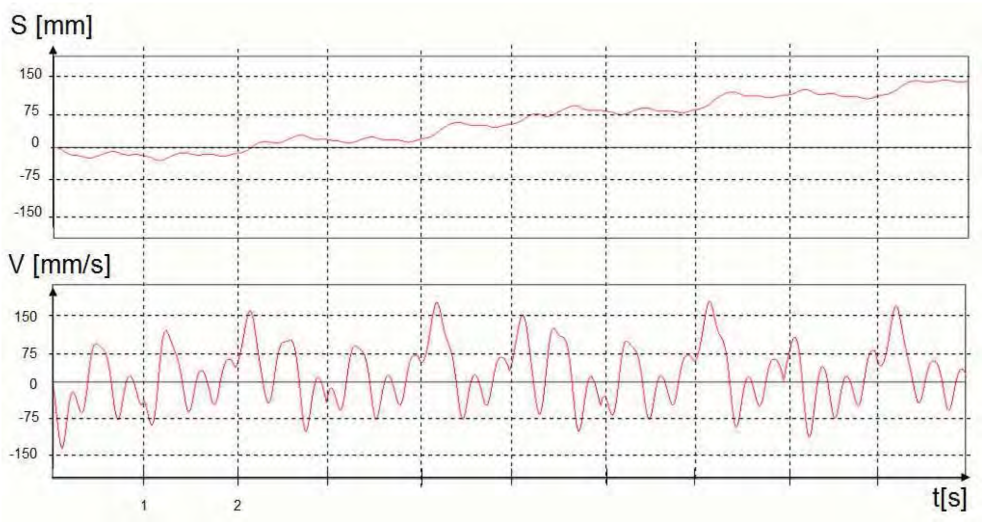

Figure 7. Response of computer simulation: the position and velocity of inductor

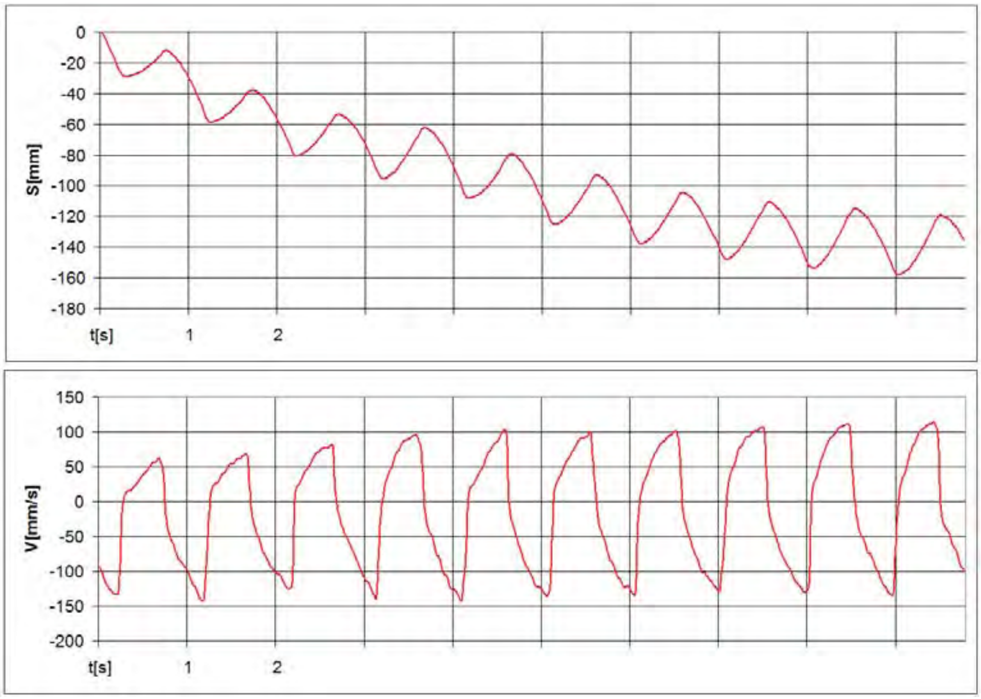

Figure 8. Response of the actual motor

In light of the result above further validation was abandoned.

5. Conclusions

The prepared and studied model does not yet reassemble actual motor with satisfactory precision. Most likely cause is oversimplification of coil shape as a perfect circle. Upgrading the model with elliptic shaped coils by modifying used integrals will most likely fix the issue. However it is believed that the model in its current form is worthy of further studying as it clearly expresses interesting chaotic behaviors due to nonlinearity of its equations.

References

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