Numerical simulation and experimental set-up of an axially excited spatial double physical pendulum

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Abstract: A periodically excited spatial double physical pendulum being coupled by two universal joints is studied. Damping forces and torques inside joints as well as an influence of the gravitational field are taken into account while deriving the governing ODEs of the pendulum dynamics. The work consists of modelling, simulation and experimental measurements to validate the numerical simulation of the earlier introduced mathematical model. In the experiment, kinetic excitation is realised by an non-constant periodic torque yielded by the computer-controlled servomotor. Angles of rotation of the pendulum links are measured by four encoders mounted on each of the universal joints and analysed by an originally developed acquisition software. Plans for future extensions of the mathematical model simulation and the experimental setup are discussed. Exemplary simulation as well as simulation of built pendulum behaviour showed several types of non-linear effects, including chaos, quasi-periodic and periodic dynamics.

1. Introduction

There are many examples of simple mechanical systems that exhibits complex behavior under certain conditions. Common mathematical or physical pendulum, e.g. a clock by C. Huygens [1], under typical situations, are not enough interesting in term of nonlinear dynamics. It appears here that the conditions are more important than the mechanisms itself. The pendulum became highly fascinating under specific circumstances, e.g. influence of excitation and damping.

This work shows similar approach. We consider the mathematical model of a simple three-dimensional double physical pendulums system, under specific conditions. The mechanism is physically excited by periodic torque in axial direction and linearly damped by each joint.

The results of numerical computations are presented, as well as possible applications of the original simulation program are discussed. A rich spectrum of regular and chaotic dynamics of the system is detected. In addition, some results of simulations for the ready built experimental setup are presented.

The evolution of pendulum analysis started from the measurement and experiment, e.g. Foucault's pendulum, 1851 [2], [3] or Kater's reversible pendulum [4]. Nowadays pendulum mechanisms are more often used in theoretical or practical mathematical modeling process. For example, to develop more effective vibrations absorption methods [5], [6]. There are also advanced research about the pendulum itself, including its experimental identification [7], [8] or control algorithms [9].

Multiple pendulum systems are mostly simplified either to planar vibrations [10], [11] or they concerning only mathematical pendulums [12]. Physical pendulums are examined in their multiple configuration, e.g. [13], [14] but also in some simplified forms.

1.1. The Pendulum Model

The system to be considered here is depicted in Figure 1. It is build of two cylindrical-shaped rigid bodies combined by universal joint O_2 and hung on a second universal joint O_1. This joint is externally driven and it actuates the entire mechanical system axially with either constant or non-constant angular velocity. Influence of the gravitational force is also included. The only damping force that the model contains is inside joints and it is characterized by a simple viscous damping function. Air resistance is neglected due to low velocities of vibrations and relatively large moments of inertia.

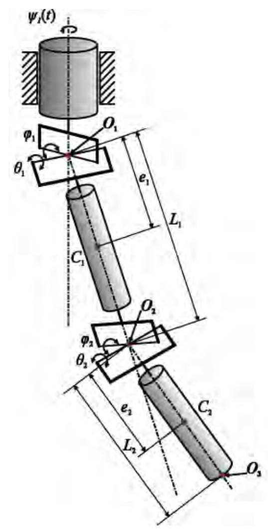

Figure 1. Coupled pendulums.

Angular positions of each universal joint's shaft have been described by three Euler angles φ_i, θ_i and ψ_i, where i corresponds to an index of each joint. The rotation matrices are found, as well as positions of each body centers. Also its linear and angular velocities and energy are defined.

Analytically determined set of nonlinear ODEs governing the pendulum dynamics follows

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{A}(\mathbf{q})\mathbf{a} + \left(\omega(t)\mathbf{B}(\mathbf{q}) + \mathbf{C}\right)\dot{\mathbf{q}} + \mathbf{r}_g(\mathbf{q}) + \omega^2(t)\mathbf{r}_\omega(\mathbf{q}) = \mathbf{0}, \tag{1}$$

where $\mathbf{q} = [\theta_1, \varphi_1, \theta_2, \varphi_2]^T$, $\mathbf{a} = \left[\dot{\theta}_1^2, \dot{\varphi}_1^2, \dot{\theta}_2^2, \dot{\varphi}_2^2, \dot{\theta}_1\dot{\varphi}_1, \dot{\theta}_1\dot{\theta}_2, \dot{\theta}_1\dot{\varphi}_2, \dot{\theta}_2\dot{\varphi}_1, \dot{\varphi}_1\dot{\varphi}_2, \dot{\theta}_2\dot{\varphi}_2\right]^T$,

and **M**, **A**, **B**, **C**, $\mathbf{r}_g$, $\mathbf{r}_\omega$ devote matrices and vectors (here not defined explicitly).

Analytical *Wolfram Mathematica*® computer package has been carried out, during process of derivation of equation (1). Full form results are too large and couldn't be simplified enough to show in this paper.

In the study, simple model of viscous damping of joints is assumed in form of:

$$\mathbf{M}_d = \left[M_{d\theta_1} - M_{d\theta_2}, M_{d\varphi_1} - M_{d\varphi_2}, M_{d\theta_2}, M_{d\varphi_2}\right]^T, \tag{2}$$

where M_i are corresponding damping torques proportional to the angular velocities.

Angular velocity of the axial excitation of first joint is as follows:

$$\dot{\psi}_1 = \omega(t) = \omega_0 + q\sin(\Omega t), \tag{3}$$

where ω_0 is a constant part of velocity [rad/s], q is the amplitude [N·m] and Ω states for frequency [rad/s].

Unlike experimental setup, the mathematical model does not take into account impacts caused by the mechanical limits of real double spatial pendulum rotation yet. That is why the full comparison of the theoretical model and experimental measurements can be performed for low amplitudes only, smaller than about 90 degrees. However, this does not prevent to achieve some really interesting simulations results based on the experimental setup parameters (not measurement), without angle limits. Measurement data will be analysed after including impacts model to the equations of the motion, increase damping in the experiment to prevent impact and/or prepare the setup to hold out frequent impacts.

2. Numerical computations

Results presented in this paper concerns the following fixed parameters (see Figure 1) presented in Table 1. Values on the left are theoretical ones, chosen for the purpose of exemplary simulations. Two columns on the right presents simulation parameters that corresponds to the measured values of built experimental setup.

Table 1. Numerical computation parameters.

	simulation example		experiment	
	first joint	second joint	first joint	second joint
weight of the pendulums [kg]	$m_1 = 0.5$	$m_2 = 0.5$	$m_1 = 3.87$	$m_2 = 2.12$
Length [m]	$L_1 = 0.2$	$L_2 = 0.2$	$L_1 = 0.22$	$L_2 = 0.2$
position of the mass center [m]	$e_1 = 0.1$	$e_2 = 0.1$	$e_1 = 0.11$	$e_2 = 0.039$
moments of inertia [kg·m]	$I_{x1} = 0.002$	$I_{x2} = 0.002$	$I_{x1} = 0.032$	$I_{x2} = 0.004$
	$I_{y1} = 0.002$	$I_{y2} = 0.002$	$I_{y1} = 0.025$	$I_{y2} = 0.007$
	$I_{z1} = 0.0001$	$I_{z2} = 0.0001$	$I_{z1} = 0.009$	$I_{z2} = 0.005$
viscous damping coefficient [N·s/m]	$c_1 = 0.1$	$c_2 = 0.1$	$c_1 = 0.1$	$c_2 = 0.1$

According to the *User manual* of *Wolfram Mathematica®* package, the ODEs solving algorithm is based on higher order Runge-Kutta methods with automatic step control. Results, as well as the plots, are automatically interpolated to any chosen time steps.

Every first 500 time steps of all numerical computations were ignored as transient motion and next 500 or more if needed were qualified as significant for the analysis.

2.1. Results and analysis

To find globally how much these two systems (example and experiment, see Table 1) are different, two-dimensional maps of maximum numbers of full 360º rotations were computed (Figure 2) for the same range of control parameters q and Ω, while ω_0 equals 0 rad/s.

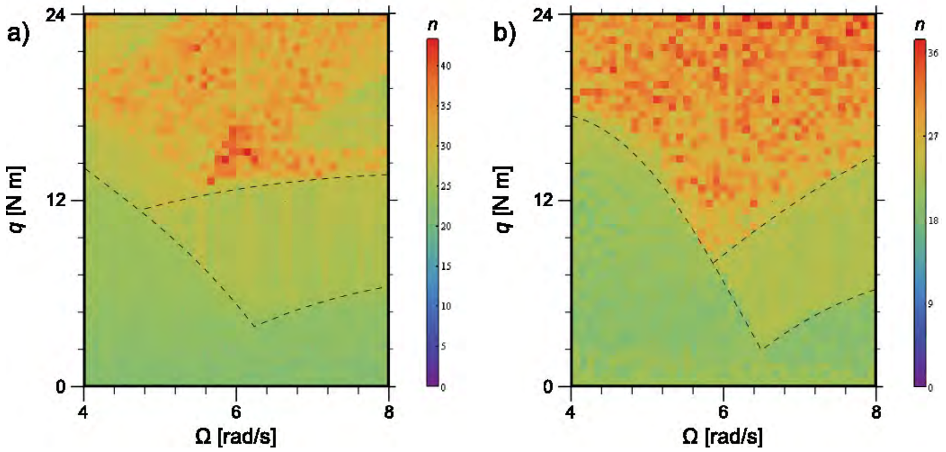

Figure 2. Number of full 360º rotations map for angle φ_1 and $\omega_0 = 0$;
a) exemplary simulation, b) simulation of built experimental setup.

On these maps one can see that there are some similarities between them, especially in the shape of the regions where the number of full rotations are low.

For more detail analysis of pendulums dynamics the bifurcation diagrams, Poincare maps and phase plots were produced. In Figure 3, three nonlinear phenomena are presented. Amplitude of excitation q was set to 12 N·m to cover all three regions of different value of full rotations (see Figure 2).

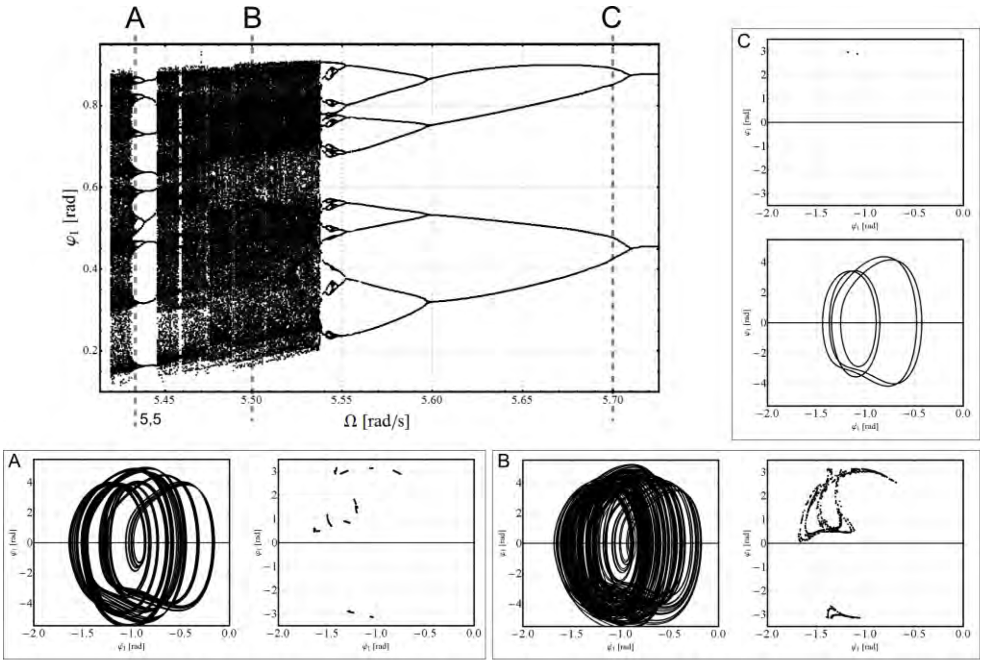

Figure 3. Bifurcational diagram regarding angle φ_1 in range $\Omega \in \langle 5.42, 5.72 \rangle$ rad/s with step 0.0006 rad/s for $\omega_0 = 0$ rad/s and $q = 12$ N·m and three sets of phase trajectories and Poincare maps corresponding three different nonlinear phenomena (for exemplary system parameters).

While changing value of the control parameter Ω, several types of nonlinear behaviour can be observed. Simple periodic vibrations, e.g. for $\Omega = 5.7$ rad/s (see Figure 3C), quasi-periodic vibrations, e.g. $\Omega = 5.5$ rad/s (see Figure 3A) or wide window of chaotic movement for Ω around 5.5 rad/s (see Figure 3B).

Similar analysis were performed for the parameters that corresponds to built experimental setup. This time, the system also showed a number of interesting nonlinearities, depicted in Figure 4.

During the simulation process, it appeared that the system is less sensitive to exhibit nonlinear behaviours than the exemplary one.

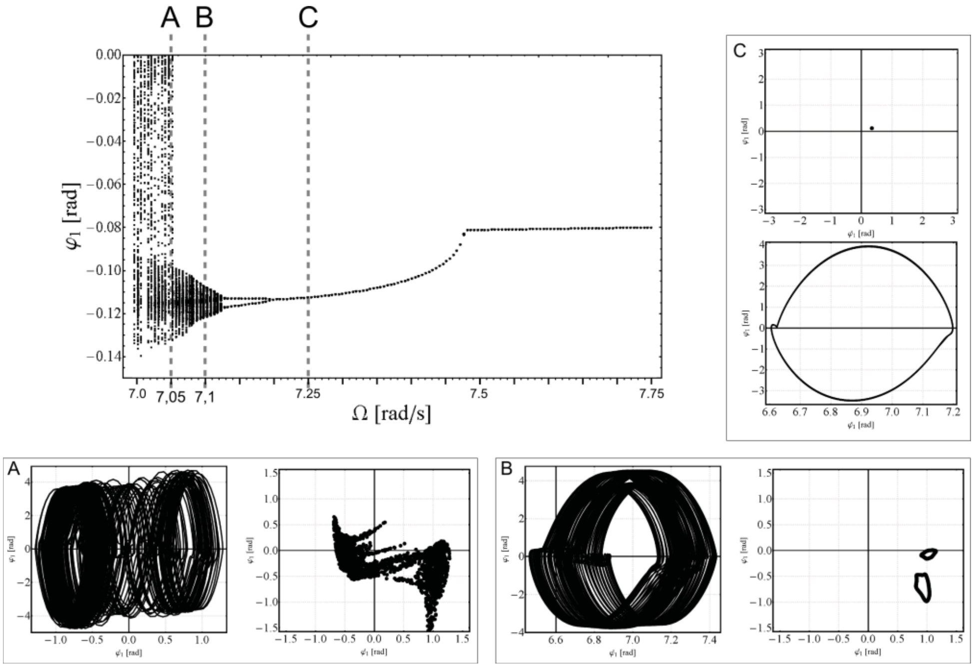

Figure 4. Bifurcational diagram regarding angle φ_1 in range $\Omega \in \langle 7, 8.5 \rangle$ rad/s with step 0.05 rad/s for ω_0 = 0 rad/s and q = 12 N·m and three sets of phase trajectories and Poincare maps corresponding three different nonlinear phenomena (for experimental setup system parameters).

3. The experimental setup

In Figure 5 one can see the photo and construction details of ready built experimental setup. To perform parallel measurements of values of all four angles of rotation the original control and acquisition software has been developed, in Java programming language. The program can control the parameters of the external excitation and simultaneously record each pendulums link position in real time.

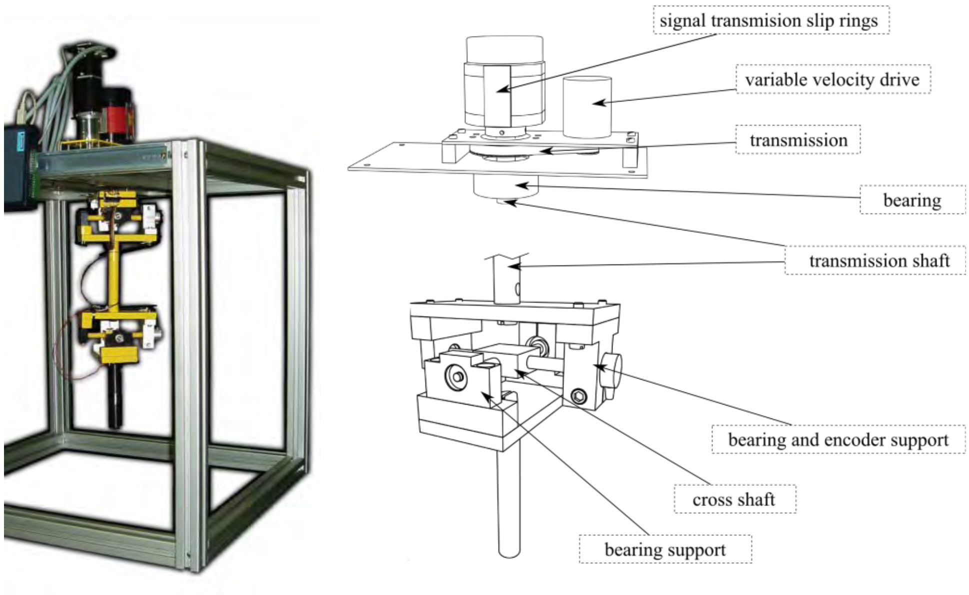

Figure 5. A part of the designed experimental stand.

The orientation of each pendulum link is measured by four precise incremental encoders and the dedicated PC acquisition card. To transmit signals between rotating pendulums equipped with the encoders and mounting frame, without the risk of the wiring damage, special slip ring is used. The external angular velocity excitation is provided by the PC-controlled servomotor.

Some preliminary measurements have been performed. Figure 6 depicts a time series measurement under constant angular velocity of excitation equals 3.7 rad/s.

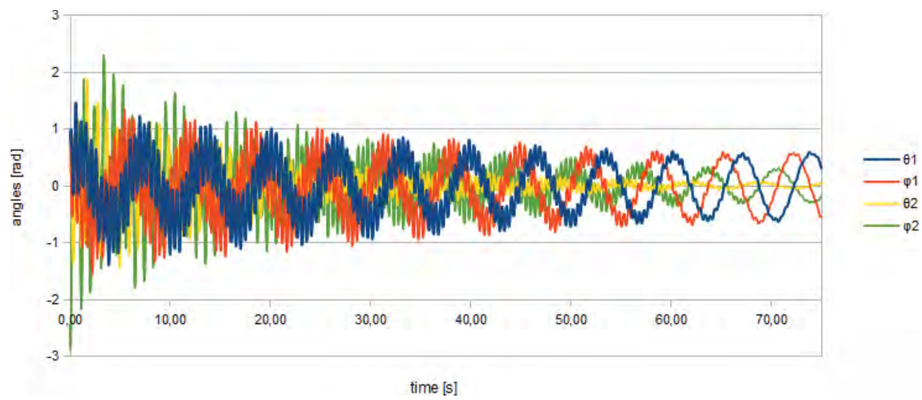

Figure 6. Time series of angular positions of all two links measured under constant angular velocity of excitation ω_0 = 3.7 rad/s, before impacts.

Presented plot of angular positions of each link shows that it takes some time before the higher frequency vibrations fades out. Afterwards, because of the constant excitation, the amplitude of vibration increases gradually to finally begin to reach its limits and note an impact. This scenario is depicted on Figure 7. Build experimental setup is not prepared for frequent collisions of its links and has to be modified to prevent its damage.

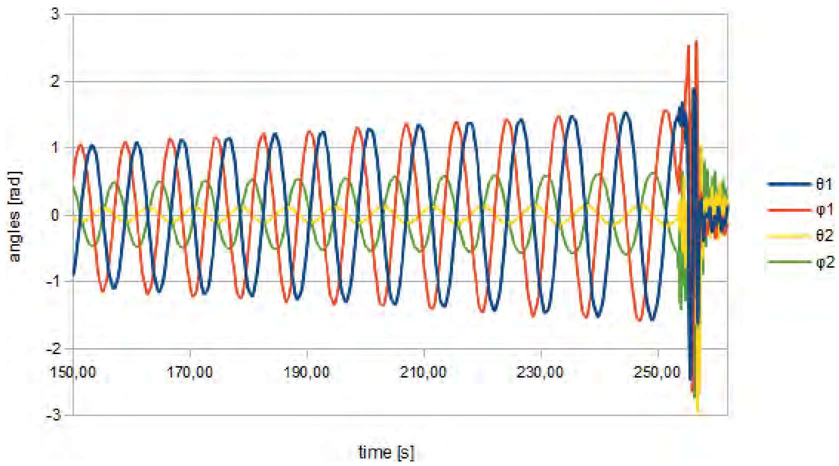

Figure 7. Time series of angular positions of all two links measured under constant angular velocity of excitation ω_0 = 3.7 rad/s, ended with impacts.

4. Conclusions

As it was presented before, pendulum either in exemplary and experimental configuration exhibits a wide spectrum of nonlinear effects. Periodic, quasi-periodic and chaotic orbits have been detected and discussed, among others. Computed full 360° rotations maps of the pendulum links shows that both configurations reveals some similarity, despite of the fact that its geometrical and mechanical parameters differs significantly.

Presented figures and its analysis confirms that performed numerical calculations can be used to simulate multiple pendulum systems.

The next step of the project is to extend the mathematical model to include impacts. Additionally, the experimental setup has to be upgrades, to modify the damping coefficient of all pendulums links and to automatically detect all impacts occurrence. That will make possibility to perform full identification of built system with no mechanical limitations.

Acknowledgments

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