Method of relaxation in theory of flexible shells

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Abstract: We study vibrations of flexible cylindrical and sectorial shells subjected to action of uniformly distributed static loads. The analyzed problems are solved using two methods: the Bubnov-Galerkin method (BGM) and the Finite Difference Method (FDM). Results validity and reliability are verified through a comparison with the results obtained by Obodan [15] in the case of non-linear static problem.

1. Introduction

Various applied load play a crucial role while estimating strength of materials involved in numerous constructions working in high temperature fields. Proper estimation of the construction strength requires the detailed analysis of the elastic-plastic material behavior, initial deflections, interaction of the construction elements or interaction of those elements with an neighborhood. Owing to complexity of the mentioned problems, the proper estimation of the construction reliability and duration requires development of suitable computational algorithms [1].

In order to solve problems regarding stability of beams, plates and shells subjected to action of a constant transversal load and taking into account geometric non-linearities, the methods of Bubnov-Galerkin in the Vlasov form as well as the Ritz and FDM methods have been applied. The mentioned numerical approaches belong to efficiently applied to solve a wide class of problems of both stationary and non-stationary mathematical physics. In the case of periodic loading, chaotic vibrations of the mentioned structural members may appear [2, 3, 4]. The mentioned approaches reduce the continuous problems to those of finite degrees of freedom [5, 6].

In order to investigate stability loss one may apply a few different criteria. Since stability loss of an arbitrary deformed object takes place in time, therefore it should be studied using various approaches of dynamics. However, majority of the stability problems of construction can be studied with static approaches, where the equilibrium states are formulated without inclusion of inertial forces, which influence on the studied system deformation.

Investigation of stability loss is carried out using a novel dynamic criterion. Namely, we define for which load buckling of equilibrium form appears. Those loads refer to the critical loads.

We omit here an overview of the fundamental works dealing with the mentioned problems, but we mention the method developed by Feodos'ev [7] regarding non-linear problems of shells.

In the latter one being originally named by Feodos'ev as the variational-step method, the system deformation is considered as a process independently on either fast or slow change of the external load. For this purpose, time is introduced artificially and equations of motion are derived. Nowadays this method refers to iteration process of finding solutions to non-linear algebraic equations, where results obtained in each time computational step are improved converging finally to the desired exact solution of the problem. In the method of relaxation a solution to PDEs is reduced to the Cauchy problem of ODEs.

2. Historical background

The proposed algorithm allows to solve a wide class of static and dynamic problems. We exhibit powerful possibilities of the given approach to solve geometrically non-linear static and dynamic problems. We consider a mechanical system subjected to action of the transversal uniformly distributed constant load over the shell surface as well as we consider the load in the form of the impulse with infinite action. Since here a crucial role plays the problem of critical static load, we briefly describe known criterion of stability proposed by numerous researchers.

Already Volmir [8] proposed the following criterion: either a fast increase of deflection corresponding to small decrease of load appears or an inflexion point of the dependence $q(w)$ ($\frac{\partial^2 q}{\partial w^2} = 0$) occurs. It has been shown that in the problems of dynamics [9] a load, where the increased process of time is responsible for achievement of the first maximum in the characteristic load-time is treated as the critical one. Kantor [10], who solved numerous problems of axially symmetric spherical shells using the Ritz method, proposed the following dynamic criterion responsible for the beam buckling. It occurs if in the shell center its deflection achieves $K \cong 2\bar{f}$, where $\bar{f} = \frac{f}{h}$, and f denotes deflection, whereas h is the shell thickness.

In references [11,12] different criteria are proposed. Namely, the system transits into a new dynamic state with the corresponding zero velocity. It can be explained in the following manner. In the beginning the inertial forces act against the external load, and after transition through zero they change sign and support action of the external load. It means that in certain time instant the beam center velocity achieves zero, and then the sudden change of deflection occurs. In reference [13] the time instant is taken as the stability loss criterion, where the displacements of an elastic body changes without a change of the associated accelerations and velocities. In some works the problem of dynamical stability loss is reduced to a quasi-dynamical problem. Owing to this approach, the pre-critical stress of the middle shell process is analyzed via static approaches. The so called "freezing" in time is applied in engineering of complex constructions. There are also works, where an dynamic criterion of the stability loss is matched with occurrence of plastic deformations of shell structures.

In reference [14] arcs were investigated, and their buckling process was characterized by two different mechanisms. In the case of direct buckling mechanism an unstable construction state was realized via symmetric forms. In the case of indirect buckling, the system lost its stability via non-symmetric forms. Since the system stability via symmetric and non-symmetric forms is qualitatively different, one may expect two different dynamic criterion of the stability loss.

In this work by a critical load we mean limiting load values or the point of inflexion of the dependence $w_{\max}(q)$. In what follows we are going to investigate critical loads acting on axially symmetric spherical and conical shells, a closed cylindrical shell as well as on a spherical sectorial shell.

3. Shallow closed cylindrical

We study shallow shells, i.e. objects in R^3 with the associated curvilinear co-ordinates $\bar{x}, \bar{y}, \bar{z}$, introduced in the following manner. In the shell body the middle surface $\bar{z} = 0$ is fixed; axes ox and oy overlap with main shell curvutures, whereas axis oz shows curvuture surface origin (Figure 3.1). In the given co-ordinates the shell is defined as follows: $\Omega = \left\{ \bar{x}, \bar{y}, \bar{z} \,/\, (\bar{x}, \bar{y}, \bar{z}) \in [0, \bar{a}] \times [0, \bar{b}] \times \left[-\bar{h}/2, \bar{h}/2 \right] \right\}$, were dimensional quantities are denoted by bars.

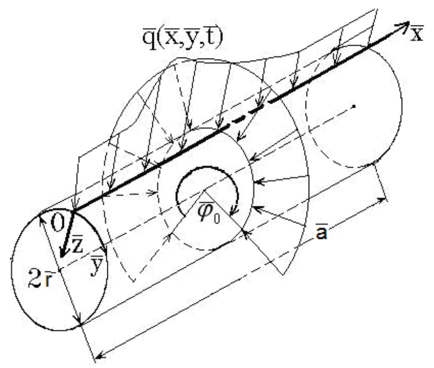


Fig. 3.1. Computational scheme of a cylindrical shell.

The governing non-linear dynamics of the shell shown in Figure 3.1 is obtained using the following hypotheses: the shell is one-layer, the shell's material is isotropic, homogenous and elastic and it satisfies the Kirchhoff-Love hypotheses. We assume that the length of the shell fiber along shell's thickness remains unchanged [8].

Therefore, in the non-dimensional form, equation of motion of the shell element as well as the deformation compatibility equations have the following non-dimensional forms:

$$\begin{gathered}
\left[\frac{1}{\lambda^2}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial x^2}+\lambda^2\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial y^2}+2(1-\mu)\frac{\partial^2 w}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}+\mu\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\cdot)}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\cdot)}{\partial x^2}\right)\right]- \\
-\nabla_k^2 F - L(w,F) + M\cdot q(t) - \left(\frac{\partial^2 w}{\partial t^2}+\varepsilon\frac{\partial w}{\partial t}\right)=0, \\
\left[\left(\lambda^2\frac{\partial^2 F}{\partial y^2}-\mu\frac{\partial^2 F}{\partial x^2}\right)\frac{\partial^2(\cdot)}{\partial y^2}+\left(\frac{1}{\lambda^2}\frac{\partial^2 F}{\partial x^2}-\mu\frac{\partial^2 F}{\partial y^2}\right)\frac{\partial^2(\cdot)}{\partial x^2}+2(1+\mu)\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2(\cdot)}{\partial x\partial y}\right]+ \\
+\nabla_k^2 w + \frac{1}{2}L(w,w)=0.
\end{gathered} \tag{3.1}$$

The following relations hold between dimensional and non-dimensional quantities:

$$w = h\overline{w}, \quad F = Eh^2\overline{F}, \quad t = t_0\overline{t}, \quad \varepsilon = \overline{\varepsilon}/\tau, \quad x = L\overline{x}, \quad y = R\overline{y}, \quad k_y = \overline{k}_y\frac{h}{R^2} \quad (k_x = 0),$$

$q = \overline{q}\dfrac{Eh^4}{L^2R^2}$, $\tau = \dfrac{LR}{h}\sqrt{\dfrac{\rho}{Eg}}$, $M = k_y{}^2$, $\lambda = \dfrac{L}{R}$, where L and $R = R_y$ correspond to the shell length and radius, respectively. In addition, we have: t - time, ε -damping coefficient, $\mu = 0.3$, $q(x,y,t)$ – transversal load. One of the following boundary conditions is taken:

1. Moving clamping

$$w = 0; \frac{\partial w}{\partial x} = 0; F = 0; \frac{\partial F}{\partial x} = 0 \quad \text{for } x = 0;1.$$

$$w = g(x,y,t); \frac{\partial w}{\partial y} = p(x,y,t); F = u(x,y,t); \frac{\partial F}{\partial y} = v(x,y,t) \quad \text{for } y = 0;\xi. \tag{3.2}$$

2. Pinned support

$$\begin{gathered}
w = 0; \frac{\partial w}{\partial x} = 0; F = 0; \frac{\partial^2 F}{\partial x^2} = 0 \quad \text{for } x = 0;1. \\
w = g(x,y,t); \frac{\partial w}{\partial y} = p(x,y,t); F = u(x,y,t); \frac{\partial F}{\partial y} = v(x,y,t) \quad \text{for } y = 0;\xi.
\end{gathered} \tag{3.3}$$

3. Moving clamping with ribs

$$w = 0; \frac{\partial^2 w}{\partial x^2} = 0; F = 0; \frac{\partial F}{\partial x} = 0 \quad \text{for } x = 0;1 .$$

$$w = g(x, y, t); \frac{\partial w}{\partial y} = p(x, y, t); F = u(x, y, t); \frac{\partial F}{\partial y} = v(x, y, t) \quad \text{for } y = 0; \xi . \tag{3.4}$$

4. Pinned support with flexible ribs

$$w = 0; \frac{\partial^2 w}{\partial x^2} = 0; F = 0; \frac{\partial^2 F}{\partial x^2} = 0 \quad \text{for } x = 0;1 .$$

$$w = g(x, y, t); \frac{\partial^2 w}{\partial y^2} = r(x, y, t); F = u(x, y, t); \frac{\partial^2 F}{\partial y^2} = z(x, y, t) \quad \text{for } y = 0; \xi . \tag{3.5}$$

Here we take $\xi = 2\pi$ for a closed cylindrical shell. In addition, the following initial conditions are applied:

$$w\big|_{t=0} = w_0, \; \dot{w}\big|_{t=0} = 0. \tag{3.6}$$

The Bubnov-Galerkin metod (BGM)

After application of the BGM the following system of algebraic-differential equations is obtained:

$$\begin{gathered} \mathbf{G}(\ddot{\mathbf{A}} + \varepsilon\dot{\mathbf{A}}) + \mathbf{HA} + \mathbf{C_1B} + \mathbf{D_1AB} = \mathbf{Q}q(t), \\ \mathbf{C}_2\mathbf{A} + \mathbf{PB} + \mathbf{D}_2\mathbf{AA} = 0, \end{gathered} \tag{3.7}$$

where: $\mathbf{H} = \|H_{ijrs}\|$, $\mathbf{G} = \|G_{ijrs}\|$, $\mathbf{C}_1 = \|C_{1ijrs}\|$, $\mathbf{C}_2 = \|C_{2ijrs}\|$, $\mathbf{D}_1 = \|D_{1ijklrs}\|$, $\mathbf{D}_2 = \|D_{2ijklrs}\|$, $\mathbf{P} = \|P_{ijrs}\|$ – square matrices of dimensions $2 \cdot N_1 \cdot N_2 \times 2 \cdot N_1 \cdot N_2$, $\mathbf{A} = \|A_{ij}\|$, $\mathbf{B} = \|B_{ij}\|$, $\mathbf{Q} = \|Q_{ij}\|$ – matrices of dimension $2 \cdot N_1 \cdot N_2 \times 1$.

Second equation of the system (3.7) is solved regarding $\mathbf{B}$ on each of the computational steps:

$$\mathbf{B} = \left[-\mathbf{P}^{-1}\mathbf{D}_2\mathbf{A} - \mathbf{P}^{-1}\mathbf{C}_2 \right]\mathbf{A}. \tag{3.8}$$

Multiplying by $\mathbf{G}^{-1}$ and taking $\dot{\mathbf{A}} = \mathbf{R}$, the following Cauchy problem is formulated for non-linear ODEs:

$$\begin{cases} \dot{\mathbf{R}} = -\bar{\varepsilon}\mathbf{R} - \left[\mathbf{G}^{-1}\mathbf{C}_1 + \mathbf{G}^{-1}\mathbf{D}_1\mathbf{A}\right]\cdot \mathbf{B} - \mathbf{G}^{-1}\mathbf{H}\mathbf{A} + \mathbf{G}^{-1}\mathbf{Q}q(\bar{t}), \\ \dot{\mathbf{A}} = \mathbf{R}. \end{cases} \tag{3.9}$$

It is solved via the fourth order Runge-Kutta method, and the computational step in time is chosen using the Runge's rule.

We apply the method of relaxation for the closed cylindrical shells with $\lambda = 2$, and we compare our results with the solution obtained by Obodan [15] for the corresponding static problem. We consider the case of transversal external load, which location is defined by the central angle φ_0.

In order to get $q_{cr}(\varphi_0)$ we need to construct a set $\{q_i, w_i\}$ for $\forall \varphi_0 \in [0;2\pi]$, which yields the critical load q_{cr}. We study the dependence of critical load versus width of the pressure zone φ_0 (Fig. 3.2). In what follows we analyze results versus various approximations. Since the load acts on the whole length of the cylindrical shell, then a number of series terms regarding *x* does not play any role, and we take only its first term, i.e. $N_1 = 1$. We investigate a dependence of the obtained results on a number of terms of the series regarding the circled coordinate *y,* i.e. on N_2. Those dependencies are presented in Fig. 3.2 for $N_2 \geq 9$. One may observe that they are non-monotonous and exhibit wave character, and although in the beginning the mentioned characteristics are not convergent, but an increase of the approximations number N_2 yields remarkable improvement of the obtained results (Fig. 3.2).

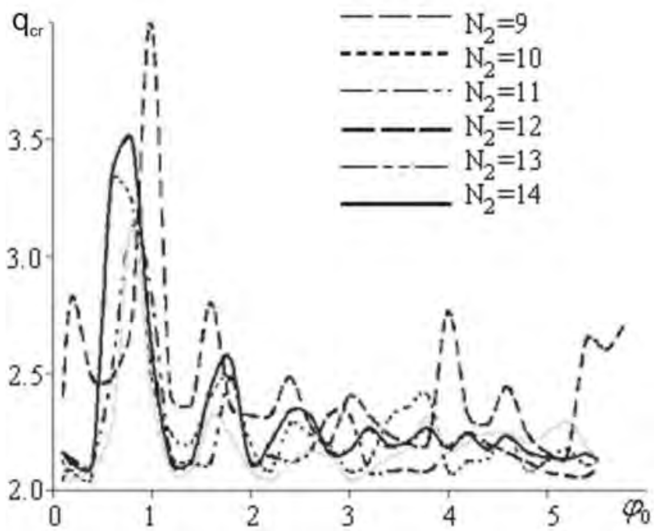
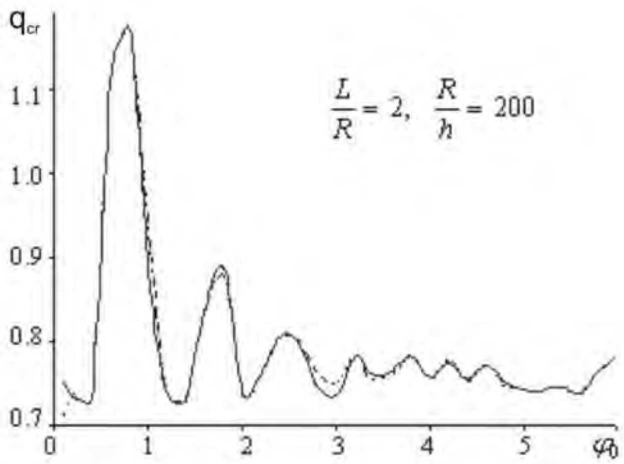


Fig. 3.2. Critical loads versus number of approximations.

The following conclusion can be formulated: In the case of non-homogeneous load, the use of low number of the series terms yields large computational errors, and the obtained results depend essentially on the number of introduced approximations. However, the situation changes qualitatively beginning from N_2=13. Namely, the dynamical properties of the cylindrical shell are stabilized, and further increase of N_2 does not improve the obtained results neither qualitatively nor quantitatively. Therefore, beginning from N_2=13 a convergent series is obtained, and all further computations are carried out for N_2=13.

Consequently, we construct the dependence of the critical loads versus width of the pressure zone $\bar{q}_{cr}(\varphi_0)$ for $N = 13$ (Fig. 3.3). Here $\bar{q}_{cr} = q_{cr}/\tilde{q}_{cr}$, where $\tilde{q}_{cr}$ refers to the classical critical value for the case of uniformly distributed external pressure, which is estimated via the Mises–Papkovitch formula [16] $\bar{q}_{cr} = 0.92\frac{R}{L}\left(\frac{h}{R}\right)^{5/2}$.



Notation:

———————— – relaxation method; ------------ – results reported in [15].

Fig. 3.3. Critical loads versus width of the pressure zone.

Dependencies $\bar{q}_{cr}(\varphi_0)$ reported by Obodan [15] for the closed cylindrical shell for $\lambda = 2$ coincide in full with the results obtained by our method (Fig. 3.4), which are shown by a solid curve. The obtained results indicate a high efficiency of the proposed method for solving static problems.

4. Shallow sector shells (FDM)

Consider a non-axially symmetric spherical shell in R^2 in the polar coordinates bounded by a contour Γ, introduced in the following way: $\bar{\Omega} = \Omega + \Gamma = \{ (r, \theta, z) \mid r \in [0, b], \theta \in [0, \theta_k], z \in [-h/2; h/2] \}$. Equations governing dynamics of shallow shells are obtained from a system of equations of the rectangular spherical shell via a transition to the polar coordinates:

$$w'' + \varepsilon w' = -\nabla^2\nabla^2 w + N(w, F) + \nabla^2 F + 4q,$$

$$\nabla^2\nabla^2 F = -\nabla^2 w - N(w, w),$$

where $\nabla^2(\cdot) = \frac{\partial^2(\cdot)}{\partial r^2} + \frac{1}{r}\frac{\partial(\cdot)}{\partial r} + \frac{1}{r^2}\frac{\partial^2(\cdot)}{\partial \theta^2}$,

$$\begin{aligned} \nabla^2\nabla^2(\cdot) &= \frac{\partial^4(\cdot)}{\partial r^4} + \frac{2}{r}\frac{\partial^3(\cdot)}{\partial r^3} - \frac{1}{r^2}\frac{\partial(\cdot)}{\partial r^2} + \frac{1}{r^3}\frac{\partial(\cdot)}{\partial r} + \\ &+ \frac{2}{r^2}\frac{\partial^4(\cdot)}{\partial \theta^2 \partial r^2} - \frac{2}{r^3}\frac{\partial^3(\cdot)}{\partial \theta^2 \partial r} + \frac{4}{r^4}\frac{\partial^2(\cdot)}{\partial \theta^2} + \frac{1}{r^4}\frac{\partial^4(\cdot)}{\partial \theta^4}, \\ N(w, F) &= \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right) + \frac{\partial^2 F}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) - \\ &- 2 \cdot \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right), \\ N(w, w) &= 2 \cdot \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) - 2 \cdot \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right]^2. \end{aligned} \tag{4.1}$$

Boundary conditions follow:

1. Pinned support of arc slices

$$w = 0, \quad \frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r} = 0, \quad F = 0, \quad \frac{\partial F}{\partial r} = 0. \tag{4.2}$$

2. Pinned support of radial slices

$$w = 0, \quad \frac{\partial^2 w}{\partial \theta^2} = 0, \quad F = 0, \quad \frac{\partial^2 F}{\partial \theta^2} = 0. \tag{4.3}$$

3. Sliding clamping of arc slices

$$w = 0, \quad \frac{\partial w}{\partial r} = 0, \quad F = 0, \quad \frac{\partial F}{\partial r} = 0. \tag{4.4}$$

4. Sliding clamping of radial slices

$$w = 0, \quad \frac{\partial w}{\partial \theta} = 0, \quad F = 0, \quad \frac{\partial^2 F}{\partial \theta^2} = 0. \tag{4.5}$$

Initial conditions are as follows:

$$w = f_1(r,\theta) = 0, \quad w' = f_2\ (r,\ \theta) = 0 \ \text{ for } t = 0\,. \tag{4.6}$$

Finite Difference Method

In order to reduce the continuous system governed by (4.1)–(4.6) to a lumped system by the FDM with the approximation $O(\Delta^2)$ versus spatial coordinates r and θ (Fig. 4.1), the following difference operators are applied:

$$\begin{gathered}-\Lambda(\Lambda w)+\Lambda_{rr}w(\Lambda F+\Lambda_{rr}F)+\Lambda_{rr}F(\Lambda w+\Lambda_{rr}w)-\\ 2\cdot\Lambda_{r\theta}w\Lambda_{r\theta}F+\Lambda F+4q_i=(w_{tt}+\varepsilon w_t)_{i,j},\\ \Lambda(\Lambda F)=-\Lambda_{rr}w(\Lambda w+\Lambda_{rr}w)+(\Lambda_{r\theta}w)^2-\Lambda w,\end{gathered} \tag{4.7}$$

where

$$\Lambda(\cdot)=\Lambda_{rr}(\cdot)+\Lambda_r(\cdot),\ \Lambda_r(\cdot)=\frac{1}{r_i^2}(\cdot)_r\,,\ \Lambda_{rr}(\cdot)=(\cdot)_{rr}\,,\ \Lambda_{r\theta}(\cdot)=-\frac{1}{r_i^2}(\cdot)_\theta+\frac{1}{r_i}(\cdot)_{r\theta}\,,$$

$$\Lambda_{rr}(\cdot)=\frac{1}{\Delta^2}\left[(\cdot)_{i+1}-2(\cdot)_i+(\cdot)_{i-1}\right],\ \Lambda_r(\cdot)=\frac{1}{2\cdot\Delta\cdot r_i^2}\left[(\cdot)_{i+1}-(\cdot)_{i-1}\right].$$

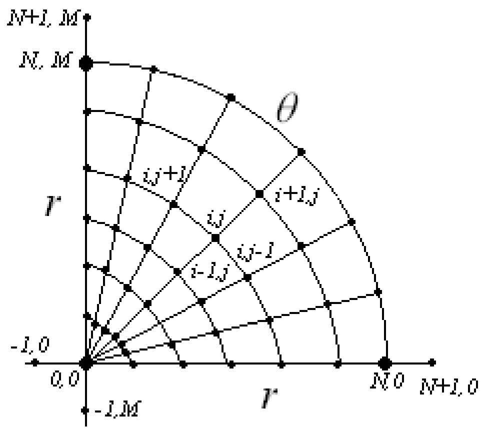


Fig. 4.1. Mesh of a sectorial shell

Boundary conditions:

1. Pinned support of arc slices

$$w_{N,j}=0\,,\ \ \Lambda_{rr}w-\frac{\nu}{b}\Lambda_r w=0\,,\ \ F_{N,j}=0\,,\Lambda_r w=0\,,\quad j=1,\ldots,M-1; \tag{4.8}$$

2. Pinned support of radial slices

$$w_{i,j} = 0, \ \Lambda_{\theta\theta} w = 0, \ F_{i,j} = 0, \Lambda_{\theta\theta} F = 0, \quad j = 0, M; \ i = 0, ..., N; \tag{4.9}$$

3. Sliding clamping of arc slices

$$w_{N,j} = 0, \ \Lambda_r w = 0, \ F_{N,j} = 0, \Lambda_r F = 0, \quad j = 1, ..., M-1; \tag{3.10}$$

4. Sliding clamping of radial slices

$$w_{i,j} = 0, \ \Lambda_{\theta\theta} w = 0, \ F_{i,j} = 0, \Lambda_{\theta\theta} F = 0, \quad j = 0, M; \ i = 0, ..., N. \tag{4.11}$$

System of equations (4.7) – (4.11) should be supplemented by conditions to be satisfied in the shell cusp and the matching conditions. In majority of the cases it is assumed that a shell has a circle hole of small dimension in its casp, and this assumption does not influence computational results essentially. In this work while solving non-symmetric problems for $\theta = 2\cdot\pi$, the approximating functions in the point $r = 0$ are interpolating by the Lagrange formula of the second order. We have

$$f_{0,j} = 3\cdot f_{1,j} - 3\cdot f_{2,j} + f_{3,j}, \tag{4.12}$$

where: $f_{i,j} = f(r_i)_j$, $r_i = i\cdot h \ (i = 0, 1, 2, 3)$, $0 \le j \le M-1$, and h is the distance between the nodes of interpolation. In the case of a point lying out of the contour the following symmetry condition holds

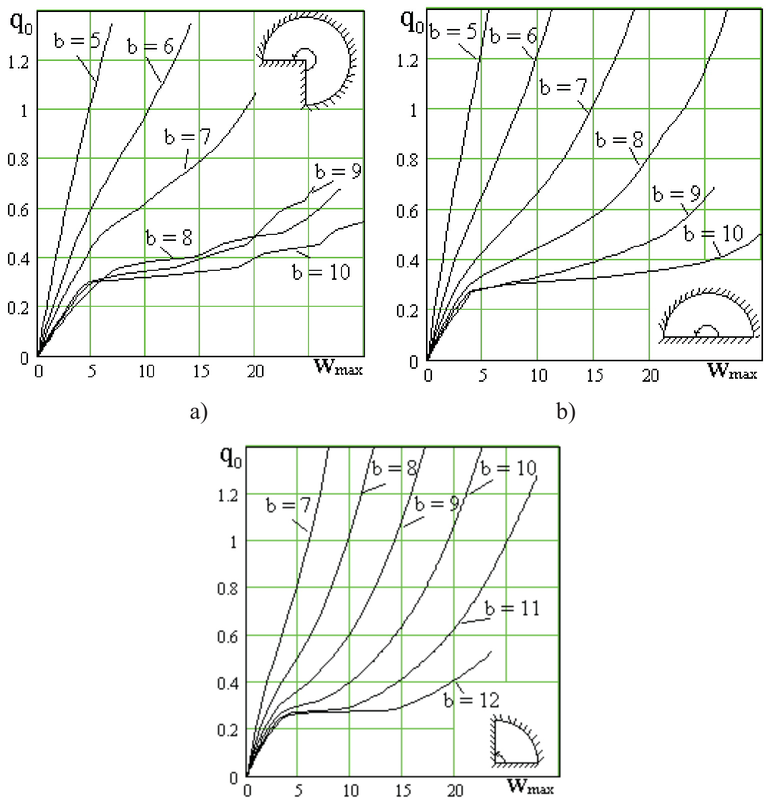
$$f_{-1,j} = f_{1,j}, \text{ for } 0 \le j \le M-1. \tag{4.13}$$

Matching conditions for non-axially symmetric problems $\theta = 2\cdot\pi$ follow:

$$w_{i,j} = w_{i,M+j}, \ F_{i,j} = F_{i,M+j} \text{ for } j = 0; -1, \ 0 \le i \le N-1. \tag{4.14}$$

The Cauchy problem (4.7) – (4.14) is solved via the fourth order Runge-Kutta method, where the computational step has been chosen due to the Runge rule [17]-[20].

In Figure 4.2 the dependencies $q(w_{yst})$ for sector angles $\theta_k = \frac{3}{2}\cdot\pi$ (a), for $\theta_k = \pi$ (b) and for $\theta_k = \frac{\pi}{2}$ (c) and for various shallow parameters for $\theta_k = \frac{3}{2}\cdot\pi$ and $\theta_k = \pi$ (b= 5; 6; 7; 8; 9; 10), and for $\theta_k = \frac{\pi}{2}$ for the shallow parameter b= 7; 8; 9; 10, 11, 12 (lower value of b are not used, since the sector shell behaves like a plate) are reported.

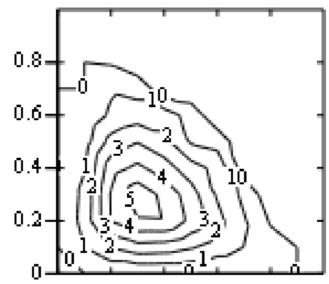
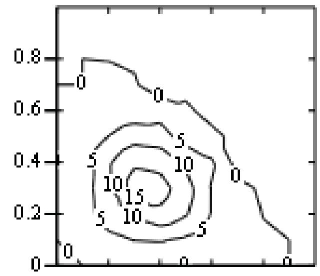
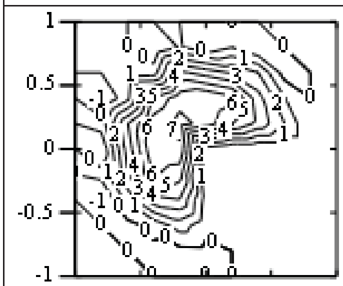
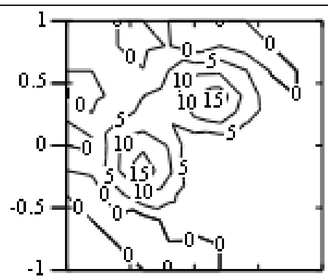


c)

Fig. 4.2. Computational results of sectorial shells analysis

The obtained graphs imply the following conclusions. Beginning from a certain value of the parameter b the critical points appear. For a shell with $\theta_k = \frac{3}{2}\cdot\pi;\ \pi;\ \frac{\pi}{2}$ and $b = 8;\ 9;\ 11$, respectively, the jump type buckling occurs. In Table 4.1 curves of equal deflections (isoclines) for all studied angles $\theta = \frac{\pi}{2};\ \pi;\ \frac{3\cdot\pi}{2}$ are shown. In what follows we compare the curves behavior for the critical and post-critical load q_0 for different θ and b. For $\theta = \frac{\pi}{2}$ and $\theta = \pi$ and for arbitrary shallow parameter the obtained results coincide, and the maximum deflection is achieved on the intersection of the bisectrix of the angle θ and the central shell radius. For $\theta = \frac{3\cdot\pi}{2}$ and for $b = 7$ the jump-type buckling is not observed, and the system dynamics is not changed qualitatively. In Figure 4.2 (a) for the same sector angle and for the first order discontinuity occurs, which implies the qualitative changes in location of isoclines for both critical and post-critical load (see Table 4.1). For $\theta = \frac{\pi}{2}$, $b=12$ a fast shell deflection is observed but without the shell form change. For $\theta = \frac{3 \cdot \pi}{2}$, $b=10$ for the critical load, the maximum shell deflection occurs on the middle of the bisectrix, whereas in the case of post-critical load two zones of maximum deflection appear, being symmetric with respect to the angle bisectrix.

Table 4.1. Computational results of sectorial shells (isoclines)

$\theta = \frac{\pi}{2}$, $b=12$, $q_0 = 0.245$, $q_0 = 0.246$	
	
$\theta = \frac{3 \cdot \pi}{2}$, $b=10$, $q_0 = 0.27$, $q_0 = 0.28$	
	

5. Concluding remarks

In this work the relaxation method has been applied to study flexible cylindrical and sectorial shells. The Bubnov-Galerkin and the finite difference approaches allowed to reduce the problems to the algebraic-ordinary differential equations. A few interesting non-linear phenomena exhibited by the analyzed continuous systems have been reported.

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