Analysis of dynamics of a non-ideal system based on decomposition of the equations of motion

Jan Awrejcewicz, Roman Starosta, Grażyna Sypniewska-Kamińska

Abstract: The dynamical response of a nonlinear 3 degrees-of-freedom (DOF) system subjected to non-ideal excitation is investigated in the paper. The excitation is said to be a non-ideal when the forces depend on the motion of the system. Such a source is described by its own differential equations and therefore, total number of DOF increases by one. In the considered system the role of non-ideal source is played by a DC motor with eccentrically suspended mass which generates a torque the magnitude of which depends on the angular velocity. During operation of the system, the general coordinate assigned to the non-ideal source is growing rapidly as a result of rotation. The main idea of the paper is to carry out the decomposition of the equations of motion in a way to extract the solution directly related to rotation of the unbalanced rotor. The remaining part of the solution describes pure oscillations depending on the dynamical behavior of the whole system. The equations of motion, decomposed in this way, have been solved numerically. The influence of selected system parameters on its dynamic behavior has been studied. The analyzed system may be considered as a good example for several engineering applications.

1. Introduction

The behavior of the mechanical systems subjected to the external periodically changing excitation belongs to the classical problems of multi-body dynamics. When the excitation is not influenced by the response, it is said to be an ideal loading. The problems similar to that are widely discussed in the literature. In the real problems the motion of the system, less or more, affects the source of energy, especially in the neighborhood of resonance. It is caused by the limited power supply to the external loading. Such source of energy is called non-ideal. The problem of non-ideal vibrations of multi-body systems leads to the sophisticated mathematical description, especially when nonlinearities appear. In that case the equation, which describes how the energy source supplies energy to the system, should be added. It causes increase of the system dimension. Therefore the non-ideal system has one more DOF than the adequate ideal one.

There are few papers concerning that phenomena. The first remarks about non-ideal vibrations was published by Sommerfeld [1], and the first book entirely devoted to that problem by Kononienko [2]. The overview of the investigations in that field is described by Balthazar et all in [3]. In that paper authors reviewed main properties of non-ideal vibrating systems. Among other the so-called Sommerfeld effect ([3,4]) connected with jump phenomena near the resonance is discussed,. That phenomenon suggests that the vibrational response provides an energy sink. The parametric resonance in the non-ideal system with DC motor was analyzed in [4,5]. Authors of the paper [4] used an asymptotic approach to investigate the behavior of the system near internal resonance.

The system investigated in the paper is presented in Figure 1. The electric DC motor with eccentrically mounted rotor is assumed to be a non-ideal source of vibrations. In that case the additional DOF responsible to its rotations occurs. The generalized co-ordinate describing motion of the rotor grows in time, and hence the whole process cannot be considered as vibrational. The new idea proposed in the paper consists in decomposition of the equation related to the rotor, and separation of the rotations and vibrations. After that operation the new set of equations of motion is derived, where all the generalized co-ordinates describe the vibrations.

2. Formulation of the problem

Let us consider the planar motion of the system consisting of DC motor of mass m_3 with unbalanced rotor of mass m_0 being supported on the rigid body of mass m_2. The eccentricity of the rotor is r_e. The support is, in turn, connected to the basis via visco-elastic bonds of the elastic and viscotic constants k and c, respectively. Length of the non-stretched spring is L_0. In addition, the mathematical pendulum of mass m_1 and length l is suspended to that rigid body. The studied system with the dimensions, co-ordinate axes and some other notation is shown in Figure 1.

The generalized co-ordinates of the system are $Z(t)$, $\Phi(t)$ and $\Lambda(t)$. The kinetic and potential energy written in the Cartesian stationary co-ordinate system are as follows

$$T = \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) + \frac{1}{2}m_3\left(\dot{x}_3^2 + \dot{y}_3^2\right) + \frac{1}{2}m_0\left(\dot{x}_0^2 + \dot{y}_0^2\right) + \frac{1}{2}I_0\dot{\Lambda}^2 \ , \tag{1}$$

$$V = -m_0 g\, x_0 - m_1 g\, x_1 - m_2 g\, x_2 - m_3 g\, x_3 + \frac{1}{2}kZ^2 \ , \tag{2}$$

where: $x_2(t) = \left(L_0 + Z(t) + h/2\right)$, $y_2 = 0$ are co-ordinates of the movable support, $x_3 = x_2 - H$, $y_3 = 0$ are co-ordinates of the engine mass center, $x_1 = x_2 + L\cos\left(\Phi(t)\right)$, $y_1 = L\sin\left(\Phi(t)\right)$ are co-ordinates of the suspended pendulum mass and $x_0 = x_2 - H + r_e\cos\left(\Lambda(t)\right)$, $y_0 = r_e\sin\left(\Lambda(t)\right)$ are co-ordinates of the mass center of the eccentrically mounted rotor.

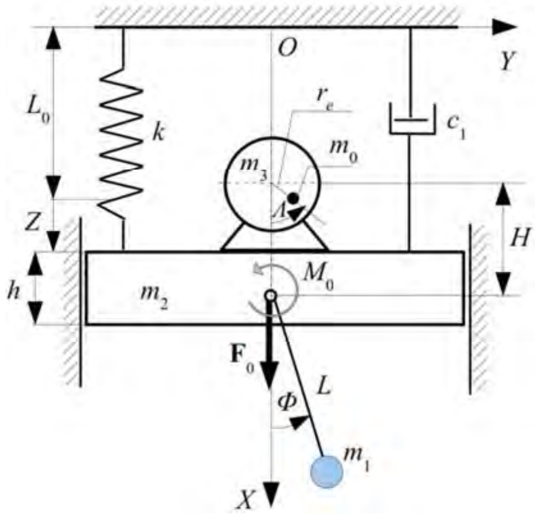

Figure 1. The investigated system.

The goal of the analysis is to study the vibrations of the system around the equilibrium position. The values of the generalized co-ordinates follow

$$Z_r = \frac{g(m_0 + m_1 + m_2 + m_3)}{k}, \Phi_r = 0, \Lambda_r = 0 . \tag{3}$$

The equations of motion have been obtained using Lagrange equations of the second type. Their non-dimensional form is:

$$\ddot{\tilde{z}} + \tilde{z} + c_1\dot{\tilde{z}} - \beta_e\mu_0\dot{\tilde{\alpha}}^2\cos\tilde{\alpha} - \mu_1\dot{\tilde{\varphi}}^2\cos\tilde{\varphi} - \beta_e\mu_0\ddot{\tilde{\alpha}}\sin\tilde{\alpha} - \mu_1\ddot{\tilde{\varphi}}\sin\tilde{\varphi} - f_1\cos(p_1\tau) = 0 , \tag{4}$$

$$\ddot{\tilde{\varphi}} + w_2^2\sin\tilde{\varphi} + c_2\dot{\tilde{\varphi}} - \ddot{\tilde{z}}\sin\tilde{\varphi} - f_2\cos(p_2\tau) = 0 , \tag{5}$$

$$\ddot{\tilde{\alpha}} + w_3^2\sin\tilde{\alpha} - u_1 + u_2\dot{\tilde{\alpha}} - \frac{w_3^2}{w_2^2}\ddot{\tilde{z}}\sin\tilde{\alpha} = 0 , \tag{6}$$

where $\tilde{z} = Z / L$, $L = L_0 + Z_r$, $\beta_e = r_e / L$, $\mu_1 = m_1 / m_c$, $\mu_0 = m_0 / m_c$, $c_1 = C_1 / m_c\omega_1$, $c_2 = C_2 / L^2 m_1\omega_1$, $w_1 = \omega_2 / \omega_1$, $w_3 = \omega_3 / \omega_1$ $\omega_2 = \sqrt{g / L}$, $\omega_3 = \sqrt{g m_0 r_e / I_s}$, $I_s = I_0 + m_0 r_e^2$ $\omega_1 = \sqrt{k_1 / m_c}$, $p_1 = \Omega_1 / \omega_1$, $p_2 = \Omega_2 / \omega_1$, $f_1 = F_0 / L m_c\omega_1^2$, $f_2 = M_0 / L^2 m_1\omega_1^2$, and dimensionless time $\tau = t\omega_1$. Now $\tilde{z}$, $\tilde{\varphi}$ and $\tilde{\alpha}$ are functions of τ.

The equations (4) – (6) are supplemented by the initial conditions:

$\tilde{\varphi}(0) = \varphi_0, \dot{\tilde{\varphi}}(0) = \psi_0, \tilde{z}(0) = z_0, \dot{\tilde{z}}(0) = v_0, \tilde{\alpha}(0) = \lambda_0, \dot{\tilde{\alpha}}(0) = \beta_0$.

According to commonly used linear model of a D.C. motor torque/speed characteristics, the torque depends linearly on the angular speed $\dot{\tilde{\alpha}}$ and is equal to $u_1 - u_2\dot{\tilde{\alpha}}$, where u_1 is related to the voltage and u_2 is a constant for each model of motor considered [4].

3. Decomposition of the governing equations

The co-ordinate $\tilde{\alpha}(\tau)$ increases boundlessly in time, what is clearly shown in Figure. 2 for the chosen data included in the set SET1={f_1=0.1, f_2=0.1, p_1=1.5, p_2=0.6, c_1=0.001, c_2=0.001, β_e=0.001, μ_1=0.7, μ_0=0.1, u_1=0.05,u_2=0.05,w_2=3,w_3=0.2}, so it does not describe any vibrational process.

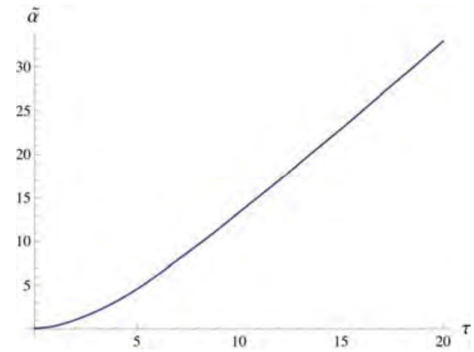

Figure 2. Time history of $\tilde{\alpha}$ for SET1.

Thus it is desirable to separate the function $\tilde{\alpha}(\tau)$ into the component describing unlimited increase and the second one corresponding to pure vibrations. We propose decomposition of that function in the following manner:

$$\alpha(\tau) = \alpha_0(\tau) + \alpha_1(\tau), \tag{7}$$

where $\alpha_0(\tau)$ satisfies the problem

$$\begin{aligned} &\ddot{\alpha}_0 - u_1 + u_2\dot{\alpha}_0 = 0, \\ &\alpha_0(0) = 0, \dot{\alpha}_0(0) = 0. \end{aligned} \tag{8}$$

Substituting (7) – (8) into (4) - (6) we obtain

$$\begin{aligned} &\ddot{z} + z + c_1\dot{z} - \beta_e\mu_0(\dot{\alpha}_0 + \dot{\alpha}_1)^2\cos(\alpha_0 + \alpha_1) - \mu_1\dot{\varphi}^2\cos\varphi \\ &- \beta_e\mu_0(\ddot{\alpha}_0 + \ddot{\alpha}_1)\sin(\alpha_0 + \alpha_1) - \mu_1\ddot{\varphi}\sin\varphi - f_1\cos(p_1\tau) = 0, \end{aligned} \tag{9}$$

$$\ddot{\varphi} + w_2^2\sin\varphi + c_2\dot{\varphi} - \ddot{z}\sin\varphi - f_2\cos(p_2\tau) = 0\,, \tag{10}$$

$$\ddot{\alpha}_1 + w_3^2 \sin(\alpha_0 + \alpha_1)_1 + u_2 \dot{\alpha}_1 - \frac{w_3^2}{w_2^2} \ddot{z} \sin(\alpha_0 + \alpha_1) = 0 \ . \tag{11}$$

The initial conditions for this set of equations are:

$$\varphi(0) = \varphi_0, \dot{\varphi}(0) = \psi_0, z(0) = z_0, \dot{z}(0) = v_0, \alpha(0) = \lambda_0, \dot{\alpha}(0) = \beta_0 \ .$$

The problem (8) describes pure rotation and can be solved analytically. The solution of it has the form:

$$\alpha_0 = -\frac{u_1\left(1 - e^{-u_2\tau} - u_2\tau\right)}{u_2^2} \ . \tag{12}$$

In the solution (12) α_0 includes transient component $u_1 e^{-u_2\tau}/u_2^2$, which disappears in time. Therefore it can be omitted, apart from the initial stage of motion, because $\alpha_0 \to \hat{\alpha}_0$, where

$$\hat{\alpha}_0 = \frac{u_1}{u_2}\tau - \frac{u_1}{u_2^2} \ . \tag{13}$$

This convergence is illustrated in Figure 3.

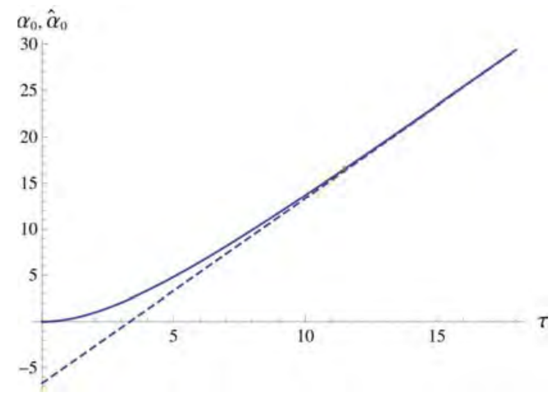

Figure 3. Asymptotic convergence α_0 to $\hat{\alpha}_0$ for SET1.

The equations obtained by introducing (12) into equations (9), (10) and (11), describe vibrations of our investigated system. Time histories of z, φ and α_1 are reported in Figures 4 – 6.

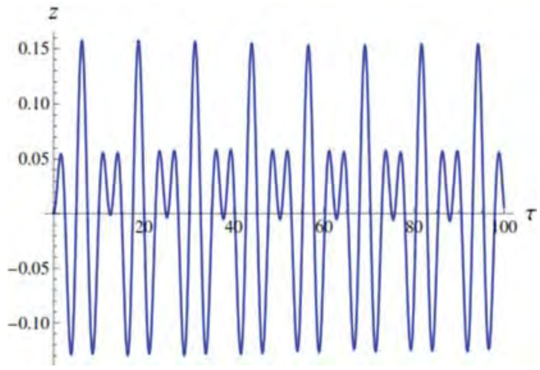

Figure 4. Time history of z for SET1.

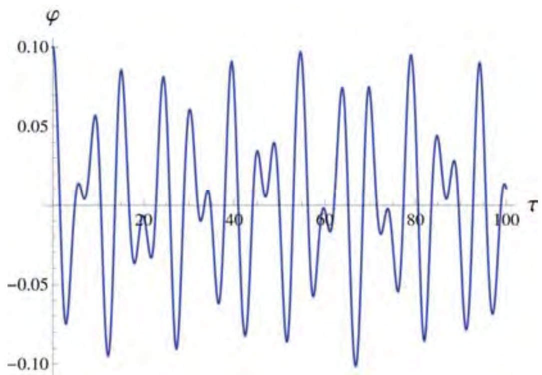

Figure 5. Time history of φ for SET1.

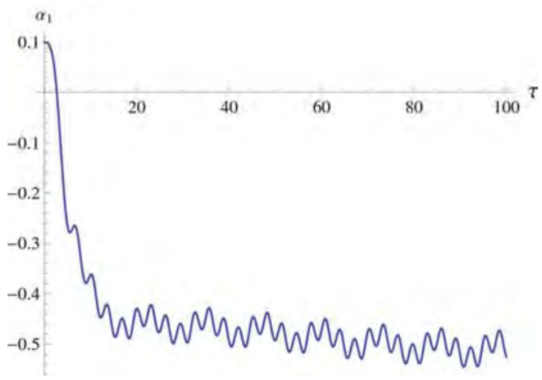

Figure 6. Time history of α_1 for SET1.

Coincidence of the solutions of the initial problems (4) – (6) and (9) – (11) (with appropriate initial conditions) is confirmed in Figure 7, where $\Delta_z = \tilde{z} - z, \Delta_\varphi = \tilde{\varphi} - \varphi, \Delta_\alpha = \tilde{\alpha} - (\alpha_0 + \alpha_1)$.

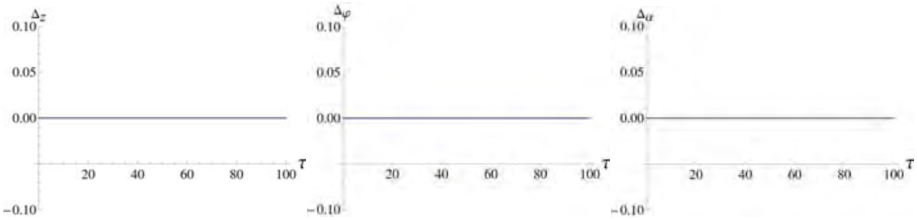

Figure 7. The identity of the solutions: $\tilde{z}$ vs z, $\tilde{\varphi}$ vs φ and $\tilde{\alpha}$ vs $(\alpha_0 + \alpha_1)$.

The conformity of the obtained solutions of the initial problems (4) – (6) and (9) – (11) is confirmed numerically for various parameters, despite the fact that the postulated form of the solution (7) is not the general solution of the problem (4) – (6).

4. Properties of the function α_1

The function α_1 describes, in principle, oscillations of the non-ideal source (beyond the transitional period), and they represent oscillations of the rotational motion. That effect is more clear for the generalized velocities. The time histories of $\dot{\alpha}_0$ and $(\dot{\alpha}_0 + \dot{\alpha}_1)$ are shown in Figure 8.

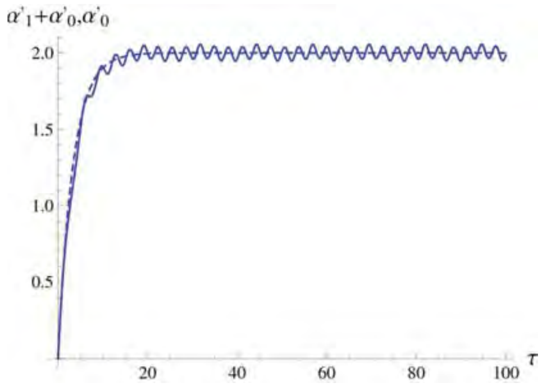

Figure 8. The time histories of $\dot{\alpha}_0$ (dashed line) and $(\dot{\alpha}_0 + \dot{\alpha}_1)$(continuous line) for SET1.

The character of the vibrations governed by α_1 strongly depends on the values of mechanical parameters. For various parameters, the oscillations can be quasi-periodic as well as chaotic. For the values of parameters collected in SET1 they are quasi-periodic. In order to detect the frequencies of α_1, the Fourier analysis is performed. Sampling covered the time interval (0, 1500) with the step equal to 0.005. In Figure 9 we can observe the mild peak referred to frequency ~1, that is a value of the linear function slope (13) of $\hat{\alpha}_0$.

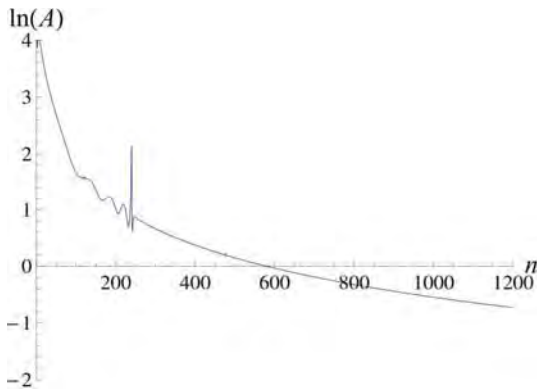

Figure 9. Discrete Fourier transform of α_1 for SET1.

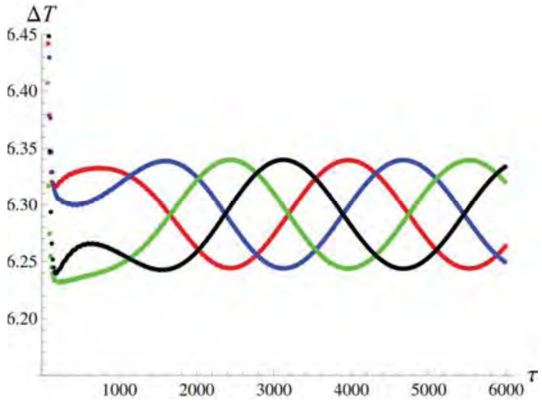

Figure 10. Time interval between two adjacent maxima of function α_1 for SET1

In Figure 10 the time interval between two neighboring maxima of function α_1 for the parameters collected in SET1 is shown. Four regular series (represented by different colors) of evaluating "period" can be observed, which corresponds to the time history presented in Figure 6. Those oscillations (in Figure 10) occur around the value equal to the slope of the linear function (13).

The analysis similar to the above, has been carried out also for the another set of parameters SET2={f_1=0.1, f_2=0.05, p_1=1.5, p_2=0.8, c_1=0.001, c_2=0.001, β_e=0.001, μ_1=0.7, μ_0=0.1, u_1=0.6, u_2=0.3, w_2=1.27, w_3=0.3}. The part of the time history of the α_1 is displayed in Figure 11. The Fourier transform for that data indicates few dominating frequencies. The greatest peak refers to the frequency ~1/2, which corresponds to the period of oscilation equal to π (see Figure 12).

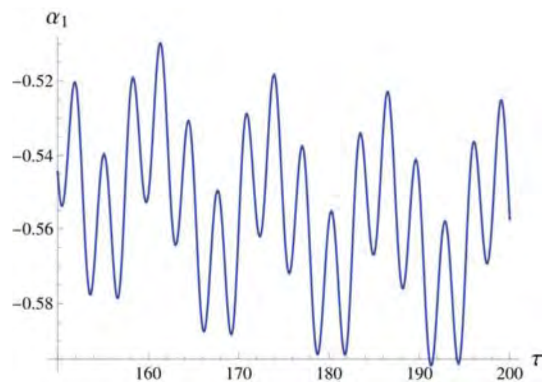

Figure 11. Time history of α_1 for SET2.

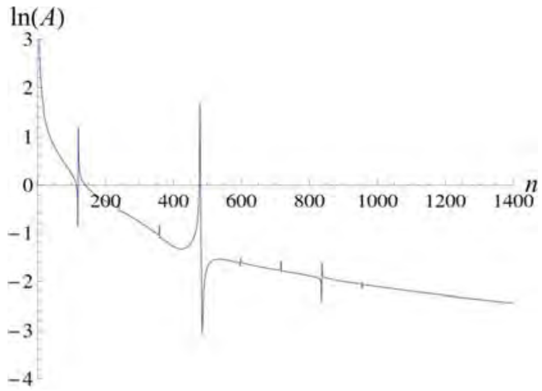

Figure 12. Discrete Fourier transform of α_1 for SET2

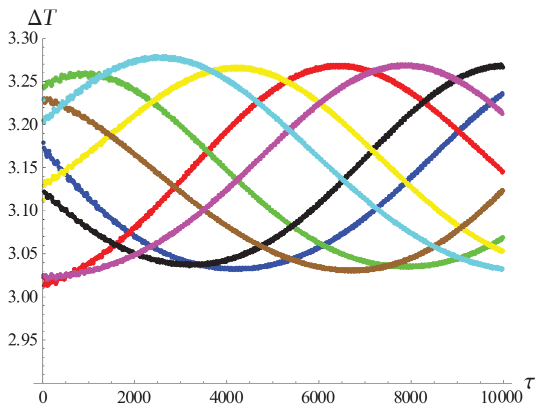

Figure 13. Time interval between two adjacent maxima of function α_1 for SET2

The curves depicting time intervals between adjacent maxima, shown in Picture 13, continuously evolve around this value. For that case eight series of evaluating "period" can be observed, which corresponds to the time history presented in Figure 11.

5. Conclusions

The nonlinear system having three degrees of freedom and excited by DC motor with unbalanced rotor, has been examined. The general coordinate assigned to the non-ideal source is growing rapidly as a result of rotation. The oscillation caused by interaction with the excited mechanical system, is imperceptible in that scale. A decomposition of the system of equations of motion has been proposed in order to separate the infinitely growing component. The proposed decomposition procedure carried means an identity in mathematical sense. Although it is not completely effective in each case, it allows to examine the oscillation of the rotor. The equations of motion, decomposed in this way, have been numerically solved. Properties of the solution describing the pure oscillation of the unbalanced rotor have been analyzed. Character of this oscillation depends strongly on the values of parameters of the system. In some cases they are quasi-periodic and their main frequency oscillates around the value equal to the slope of the asymptote of the function α_0 representing pure rotation.

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