Bifurcation and Chaos of Multi-body Dynamical Systems

Jan Awrejcewicz and G. Kudra

Abstract Triple physical pendulum in a form of three connected rods with the first link subjected to an action of constant torque and with a horizontal barrier is used as an example of plane mechanical system with rigid limiters of motion. Special transition rules for solutions of linearized equations at impact instances (Aizerman-Gantmakher theory) are used in order to apply classical tools for Lyapunov exponents computation as well as for stability analysis of periodic orbits (used in seeking for stable and unstable periodic orbits and bifurcations of periodic solutions analysis). Few examples of extremely rich bifurcational dynamics of triple pendulum are presented.

Keywords Pendulum • Impact • Bifurcation • Periodic orbit • Quasi-periodic orbit • Chaotic attractor • Lyapunov exponents • Non-smooth dynamics

1 Introduction

A single or a multiple pendulum (in their different forms) are very often studied theoretically or experimentally [1–3]. A single pendulum plays an important role in mechanics since many interesting non-linear dynamical behavior can be illustrated and analyzed using this simple system. But a single degree-of-freedom models are only the first step to understand a real behavior of either natural or engineering systems. Many physical objects are modeled by a few degrees of freedom and an attempt to investigate coupled pendulums is recently observed.

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On the other hand, it is well known that impact and friction accompanies almost all real behavior, leading to non-smooth dynamics. The example of modeling of the piston – connecting rod – crankshaft system by the use of triple physical pendulum with rigid limiters of motion is presented in the work [4].

The non-smooth dynamical systems can be modeled as the so-called piece-wise smooth systems (PWS) and they are also interesting from a point of view of their bifurcational behavior, since they can exhibit certain non-classical phenomena of non-linear dynamics [5, 6]. One of the important tools of non-linear dynamics is the linear stability theory, useful among others in the analysis of bifurcations of periodic solutions and in the identification of attractors through Lyapunov exponents. These tools are well-developed and known in the case of smooth systems. However the same tools with small modifications [6, 7] can be also used for the PWS systems. The modifications consist in the suitable transformation of the perturbation in the point of discontinuity, accordingly to the so called Aizerman-Gantmakher theory [8, 9].

In the present paper some examples of identification of attractors in the system of triple physical pendulum with the horizontal barrier are given. The system used is a special case of the more general model of triple pendulum investigated in earlier works of the authors.

2 Event Driven Model of Mechanical System with Limiters of Motion

Let us assume firstly more general case of mechanical system of n-degrees-of-freedom with vector of generalized coordinates $\mathbf{q}(t) = [q_1(t), \ldots, q_n(t)]^T$, symmetric $n \times n$ mass matrix $\mathbf{M}(\mathbf{q}, t)$ and $n \times 1$ force vector $\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t)$. The system is subjected to m rigid unilateral constraints $\mathbf{h}(\mathbf{q}, t) = [h_1(\mathbf{q}, t), \ldots, h_m(\mathbf{q}, t)]^T \geq 0$. We define a set $I = \{1, 2, \ldots, m\}$ of indices of all defined unilateral constraints h_i and the set $I_{act} = \{i_1, i_2, \ldots, i_s\}$ of indices of s constraints permanently active on a certain time interval $[t_i, t_{i+1}]$. Physically it means that the system slides along obstacles with indices from the set I_{act}.

In the case of frictionless constraints, the system on time interval $[t_i, t_{i+1}]$ is governed by the following set of differential and algebraic equations (DAEs)

$$\mathbf{M}(\mathbf{q}, t)\, \ddot{\mathbf{q}} = \mathbf{f}_{\mathbf{q}}(\mathbf{q}, \dot{\mathbf{q}}, t) + \left(\frac{\partial \mathbf{h}_{act}(\mathbf{q}, t)}{\partial \mathbf{q}^T}\right)^T \boldsymbol{\lambda}_{act},$$

$$0 = \mathbf{h}_{act}(\mathbf{q}, t), \quad 0 = \dot{\mathbf{h}}_{act}(\mathbf{q}, t) = \frac{\partial \mathbf{h}_{act}(\mathbf{q}, t)}{\partial \mathbf{q}^T}\dot{\mathbf{q}} + \frac{\partial \mathbf{h}_{act}(\mathbf{q}, t)}{\partial t} \tag{1}$$

with the following event functions determining the time instances t_{i+1}

$$\boldsymbol{\lambda}_{act} = [\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_s}]^T > 0,$$

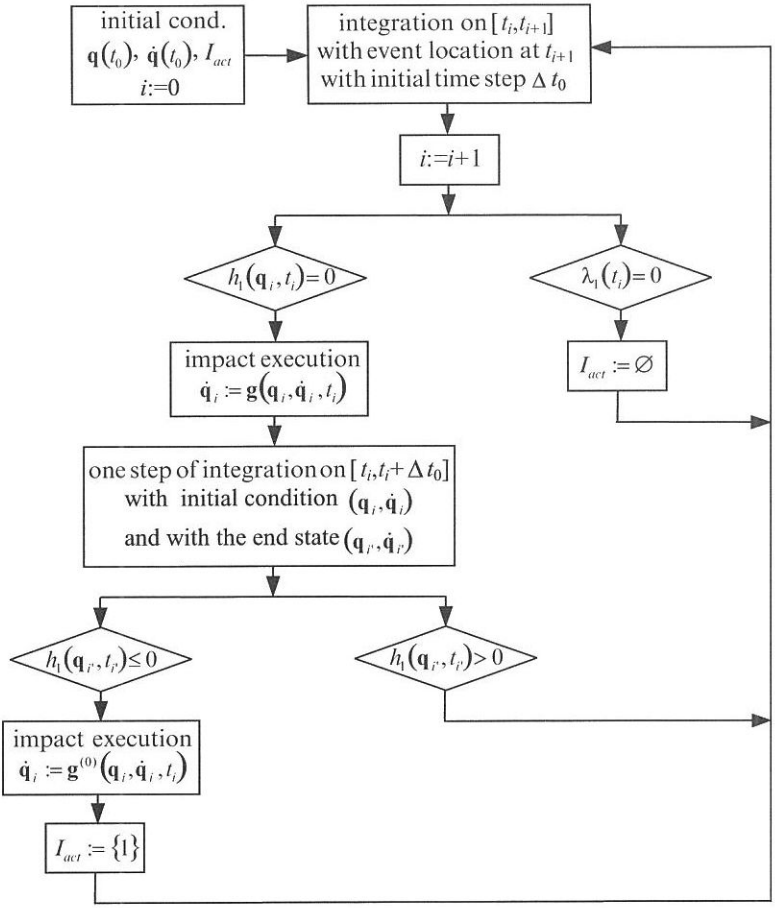

Fig. 1 Scheme for the numerical simulation of the system

$$\mathbf{h}_{inact}(\mathbf{q}, t) = \left[h_{j_1}(\mathbf{q}, t), h_{j_2}(\mathbf{q}, t), \ldots, h_{j_{m-s}}(\mathbf{q}, t)\right]^T > 0, \quad (2)$$

where $\mathbf{h}_{act}(\mathbf{q}, t) = [h_{i_1}(\mathbf{q}, t), h_{i_2}(\mathbf{q}, t), \ldots, h_{i_s}(\mathbf{q}, t)]^T$ is the vector of s constraints permanently active on $[t_i, t_{i+1}]$, $\boldsymbol{\lambda}_{act}$ is the vector of non-negative Lagrange multipliers and $\mathbf{h}_{inact}$ is the vector of $m-s$ inactive constraints, i.e. constraints which indices belong to the set $I \backslash I_{act} = \{j_1, j_2, \ldots, j_{m-s}\}$. The event t_{i+1} is determined by the use detection of zero-crossing of any component of $\boldsymbol{\lambda}_{act}$ or $\mathbf{h}_{inact}$. At time instance t_{i+1} the suitable changes in initial conditions (due to the impact) and in the set I_{act} take place and the next piece of solution $[t_{i+1}, t_{i+2}]$ is governed by the new DAEs. In this way the system has been modeled as a piece-wise smooth (PWS) DAEs.

The algorithm for the execution of changes in the system state and changes in the set I_{act} at each event time t_j, used in our numerical simulation, is presented in Fig. 1. Because of the limited space, we restrict this scheme to the simplified case, where only one constraint $h_1(\mathbf{q}, t)$ is defined ($I = \{1\}$). In the Fig. 1 the following notations are used: $\mathbf{q}_j = \mathbf{q}(t_j)$, $\dot{\mathbf{q}}_j = \dot{\mathbf{q}}(t_j)$, $t_{j'} = t_j + \Delta t_0$ and the function $\mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}, t)$ represents the impact law with the restitution coefficient e while the function $\mathbf{g}^{(0)}(\mathbf{q}, \dot{\mathbf{q}}, t)$ represents impact with the restitution coefficient equal to zero independently from the system parameters.

The applied impact model is the generalized Newton's (restitution coefficient) impact law based on the reference [5], and has the following final form for the impact with the obstacle defined by $h_i(\mathbf{q}, t) = 0$:

$$\mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}, t) = \begin{bmatrix} (\nabla_{\mathbf{q}} h_i(\mathbf{q}, t))^T \\ \begin{bmatrix} \mathbf{t}_1^T \\ \cdots \\ \mathbf{t}_{n-1}^T \end{bmatrix} \cdot \mathbf{M}(\mathbf{q}, t) \end{bmatrix}^{-1} \cdot \left(\begin{bmatrix} -e(\nabla_{\mathbf{q}} h_i(\mathbf{q}, t))^T \\ \begin{bmatrix} \mathbf{t}_1^T \\ \cdots \\ \mathbf{t}_{n-1}^T \end{bmatrix} \cdot \mathbf{M}(\mathbf{q}, t) \end{bmatrix} \dot{\mathbf{q}} + \begin{Bmatrix} -(e+1)\dfrac{\partial h_i(\mathbf{q}, t)}{\partial t} \\ 0 \\ \cdots \\ 0 \end{Bmatrix} \right), \tag{3}$$

where $\mathbf{t}_j$ are the base vectors of the subspace of the configuration space $\mathbf{q}$, tangent to the impact surface $h_i(\mathbf{q}, t)$ at the impact point. For more details on the impact model see works [3, 4, 7].

3 Linear Stability Model

For the dynamical system in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{4}$$

where $\mathbf{x} = [\mathbf{q}^T, \dot{\mathbf{q}}^T]^T$, the small perturbation of the solution is governed by the following linear equations

$$\delta\dot{x} = \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}^T} \delta x(t), \tag{5}$$

where we have assumed $\delta t = 0$ since the perturbation in time is independent from the perturbation δx ($\delta\dot{t} = 0$). The Eq. 5 are useful among others in the stability and bifurcation analysis of periodic solutions, as well as in the Lyapunov exponents calculation.

In the case of non-smooth dynamical system we cannot apply directly the linear stability theory since the Jacobian in (5) is not determined. But in the case of the PWS system the function $\mathbf{f}(\mathbf{x}) = \mathbf{f}_i(\mathbf{x})$ is sufficiently smooth on each time interval $[t_i, t_{i+1}]$ between two successive discontinuity points and the linear stability can applied using variational Eq. 5 on intervals $[t_i, t_{i+1}]$, and applying at each discontinuity point t_i special transformation rules accordingly to the Aizerman-Gantmakher theory (for $\delta t = 0$):

$$\delta x_i^+ = \frac{\partial \mathbf{g}_i\left(\mathbf{x}_i^-, t_i\right)}{\partial \mathbf{x}^T}\delta x_i^- + \left[\frac{\partial \mathbf{g}_i\left(\mathbf{x}_i^-, t_i\right)}{\partial \mathbf{x}^T}\mathbf{f}_i\left(\mathbf{x}_i^-, t_i\right) + \frac{\partial \mathbf{g}_i\left(\mathbf{x}_i^-, t_i\right)}{\partial t} - \mathbf{f}_{i+1}\left(\mathbf{x}_i^+, t_i\right)\right]\delta t_e \tag{6}$$

where

$$\delta t_e = -\frac{\dfrac{\partial event_i\left(\mathbf{x}_i^-, t_i\right)}{\partial \mathbf{x}^T}\delta x_i^-}{\dfrac{\partial event_i\left(\mathbf{x}_i^-, t_i\right)}{\partial \mathbf{x}^T}\mathbf{f}_i\left(\mathbf{x}_i^-, t_i\right) + \dfrac{\partial event_i\left(\mathbf{x}_i^-, t_i\right)}{\partial t}},$$

and where $\mathbf{x}_i^- = \lim_{t \to t_i^-}\mathbf{x}(t)$, $\mathbf{x}_i^+ = \lim_{t \to t_i^+}\mathbf{x}(t)$, $\delta x_i^+ = \lim_{t \to t_i^+}\delta x(t)$, $\delta x_i^- = \lim_{t \to t_i^-}\delta x(t)$, $\mathbf{g}_i(\mathbf{x})$ is the function representing jump in the system state $\mathbf{x}_i^+ = \mathbf{g}_i\left(\mathbf{x}_i^-\right)$ in the discontinuity point and $event_i(\mathbf{x}, t)$ is the scalar function used for detection of the discontinuity instance at t_i $\left(event_i\left(\mathbf{x}_i^-, t_i\right) = 0\right)$.

The linearized differential-algebraic equations of the system (1) are

$$\mathbf{M}(\mathbf{q}, t)\,\delta\ddot{\mathbf{q}} = \frac{\partial \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \mathbf{q}^T}\delta\mathbf{q} + \frac{\partial \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{\mathbf{q}}^T}\delta\dot{\mathbf{q}} + \frac{\partial}{\partial \mathbf{q}^T}\left(\left(\frac{\partial \mathbf{h}_{act}(\mathbf{q})}{\partial \mathbf{q}^T}\right)^T \boldsymbol{\lambda}_{act}\right)\delta\mathbf{q}$$

$$+\left(\frac{\partial \mathbf{h}_{act}(\mathbf{q})}{\partial \mathbf{q}^T}\right)^T \delta\boldsymbol{\lambda}_{act} - \left(\frac{\partial \mathbf{M}(\mathbf{q}, t)}{\partial \mathbf{q}^T}\delta q\right)\ddot{\mathbf{q}},\ 0 = \frac{\partial \mathbf{h}_{act}(\mathbf{q}, t)}{\partial \mathbf{q}^T}\delta q$$

$$0 = \dot{\mathbf{q}}^T\frac{\partial^2 \mathbf{h}_{act}(\mathbf{q}, t)}{\partial \mathbf{q}\partial \mathbf{q}^T}\delta q + \frac{\partial \mathbf{h}_{act}(\mathbf{q}, t)}{\partial \mathbf{q}^T}\delta\dot{\mathbf{q}}, \tag{7}$$

where

$$\ddot{\mathbf{q}} = \mathbf{M}(\mathbf{q}, t)^{-1}\left(\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t) + \left(\frac{\partial \mathbf{h}_{act}(\mathbf{q}, t)}{\partial \mathbf{q}^T}\right)^T \boldsymbol{\lambda}_{act}\right)$$

and where we have also assumed $\delta t = 0$.

We have applied Eq. 7 together with the transformation rules (6) in the Lyapunov exponents calculation for the mechanical system presented in the Sect. 2. Note that

Eq. 6 with the impact law $\mathbf{g}_i(\mathbf{x},t) = \mathbf{g}_i^{(0)}(\mathbf{x},t)$ with the restitution coefficient equal to zero applied in the case where the sliding motions starts (see Fig. 1), gives the perturbation $(\delta\mathbf{q}, \delta\dot{\mathbf{q}})$ consistent with the algebraic equations in (7) and the perturbation vector $\delta\mathbf{x}^+$ lies in the (2n-2)-dimensional subspace (in the case of only one constraint permanently active).

In the well-known algorithm of Lyapunov exponents computation the Gram-Schmidt reorthonormalization procedure is applied after some time of integration of variational equations. After use of this procedure to the vector of perturbations δx fulfilling 2s algebraic equations in (7) (in the case of s constraints permanently active), we obtain the new set of perturbation vectors, from which 2n-2s satisfy the algebraic equations and 2s of them do not. Then in our procedure we simply set that 2s vectors to zero vectors, obtaining the new "degenerated" set of orthonormal vectors, satisfying algebraic equations.

4 Triple Pendulum Model

Three joined stiff links coupled with viscous damping and moving on the plane are presented in Fig. 2. The system position is defined by three angles ψ_i ($i = 1,2,3$), and each of the first body is under action of constant torque q_1. The set of possible

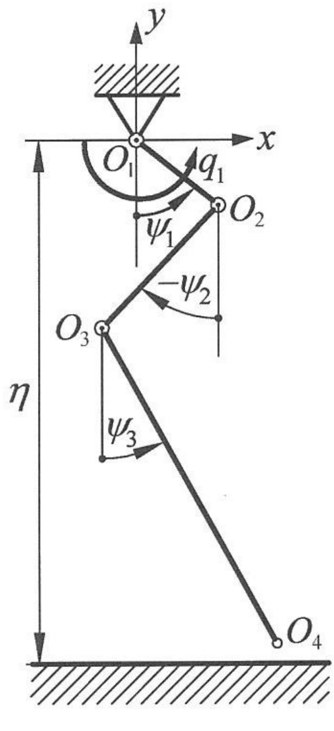

Fig. 2 Mechanical system

configurations of the system is bounded by the horizontally situated rigid and frictionless barrier. A vector of generalized coordinates is the vector of three angles $\mathbf{q} = \boldsymbol{\psi} = [\psi_1, \psi_2, \psi_3]^T$. The mass matrix, force vector and the set of algebraic equations defining rigid obstacle are as follows

$$\mathbf{M}(\mathbf{q}, t) = \mathbf{M}(\boldsymbol{\psi}) = \begin{bmatrix} 1 & \nu_{12}\cos(\psi_1 - \psi_2) & \nu_{13}\cos(\psi_1 - \psi_3) \\ \nu_{12}\cos(\psi_1 - \psi_2) & \beta_2 & \nu_{23}\cos(\psi_2 - \psi_3) \\ \nu_{13}\cos(\psi_1 - \psi_3) & \nu_{23}\cos(\psi_2 - \psi_3) & \beta_3 \end{bmatrix},$$

$$\mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{f}(\boldsymbol{\psi}, \dot{\boldsymbol{\psi}}, t) = -\mathbf{N}(\boldsymbol{\psi})\,\dot{\boldsymbol{\psi}}^2 - \mathbf{C}\,\dot{\boldsymbol{\psi}} - \mathbf{p}(\psi) + \mathbf{f}_e(\boldsymbol{\psi}, \dot{\boldsymbol{\psi}}, t), \qquad (8)$$

$$h_1(\boldsymbol{\psi}) = \eta - l_1\cos\psi_1, \quad h_2(\boldsymbol{\psi}) = \eta - \sum_{i=1}^{2} l_i\cos\psi_i, \quad h_3(\boldsymbol{\psi}) = \eta - \sum_{i=1}^{3} l_i\cos\psi_i$$

where

$$\mathbf{N}(\boldsymbol{\psi}) = \begin{bmatrix} 0 & \nu_{12}\sin(\psi_1 - \psi_2) & \nu_{13}\sin(\psi_1 - \psi_3) \\ -\nu_{12}\sin(\psi_1 - \psi_2) & 0 & \nu_{23}\sin(\psi_2 - \psi_3) \\ -\nu_{13}\sin(\psi_1 - \psi_3) & -\nu_{23}\sin(\psi_2 - \psi_3) & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}, \quad \mathbf{p}(\boldsymbol{\psi}) = \begin{Bmatrix} \sin\psi_1 \\ \mu_2\sin\psi_2 \\ \mu_3\sin\psi_3 \end{Bmatrix}, \quad \mathbf{f}_e = \begin{Bmatrix} q_1 \\ 0 \\ 0 \end{Bmatrix},$$

and where $\dot{\boldsymbol{\psi}}^2 = \left[\dot{\psi}_1^2, \dot{\psi}_2^2, \dot{\psi}_3^2\right]^T$, where l_i is non-dimensional length of i-th link, c_i is non-dimensional damping coefficient in the i-th joint while ν_{ij} and μ_i are other non-dimensional parameters of the system.

The system response is obtained numerically by the use of the Runge-Kutta integration method of the differential equations between each two successive discontinuity points (where the activity of the obstacles changes: the impact takes place or the time interval of sliding begins or ends). These points are detected by halving integration step until obtaining assumed precision. After the simulation of the system, the next step was the stability analysis of the solution in the investigated model, which in fact is piece-wise smooth (PWS) one. The classical methods and algorithms basing on the linear perturbation equations are used with the modifications taking into account the perturbations jump in the discontinuity points [9]. The numerical software for Lyapunov exponents calculation and periodic orbit stability analysis (seeking for periodic orbits and their bifurcations analysis) was developed.

For more details on modeling, relations between real and non-dimensional parameters, numerical algorithms, etc., see works [3, 4, 7].

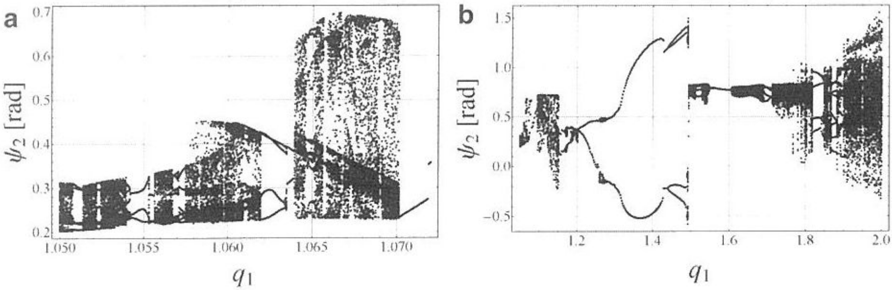

Fig. 3 Bifurcational diagrams

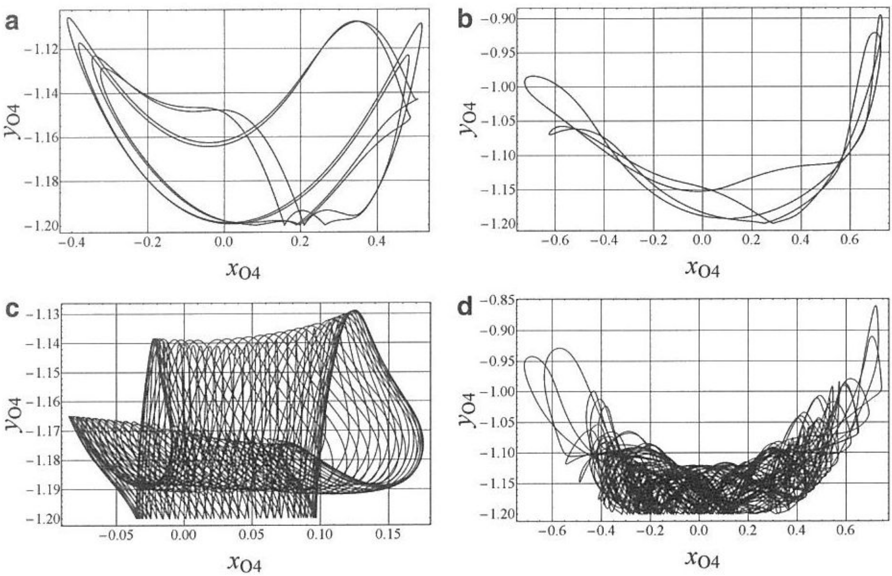

Fig. 4 Projections of periodic (**a**, $q_1 = 1.063$; **b**, $q_1 = 1.59$), quasi-periodic (**c**, $q_1 = 1.63$) and chaotic (**d**, $q_1 = 2$) attractors

5 Numerical Examples

The examples of extremely rich bifurcational dynamics of the modeled system is presented for the following non-dimensional parameters: $l_1 = O_1\ O_2 = 0.05$, $l_2 = O_2\ O_3 = 0.02$, $l_3 = O_3\ O_4 = 1$, $\eta = 1.2$ and $c_1 = c_2 = c_3 = 0.8$. The restitution coefficient is $e = 0.8$ and contact between links and obstacles is assumed to be frictionless. The externally applied torque q_1 is used as bifurcational parameter.

In Fig. 3 one can find two bifurcational diagrams where the parameter q_1 is increasing quasi-statically. In Fig. 3a the relative change of the torque is very small (about 2%) but the richness and number of bifurcational phenomena observed is

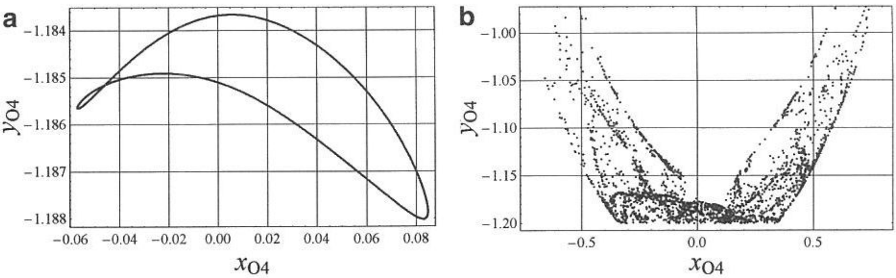

Fig. 5 Projections of Poincaré sections of quasi-periodic (**a**, $q_1 = 1.063$) and chaotic (**b**, $q_1 = 2$) attractors

Table 1 Lyapunov exponents

Figure	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	Attractor
3a	*0.00*	−0.02	−0.03	−0.24	−0.35	−5.98	Periodic
3b	*0.00*	−0.02	−0.02	−0.22	−0.47	−8.44	Periodic
3c and 4a	*0.00*	*0.00*	−0.01	−0.08	−0.54	−10.28	Quasi-periodic
3d and 4b	**0.05**	*0.00*	−0.03	−0.19	−0.52	−10.03	Chaotic

extremely large. Both bifurcational diagrams start just after disappearing of stable equilibrium position for q_1 equal about 1.01. The next Figures exhibit exemplary periodic, quasiperiodic and chaotic attractors observed on bifurcational diagrams. Fig. 4 presents trajectory projections while Fig. 5 shows corresponding Poincaré sections (performed by the use of plane $\psi_1 = 0$). The verification of a kind of each the attractor is performed by the use of Lyapunov exponents presented in Table 1.

6 Conclusions

This paper briefly reports the larger project of investigations of the flat triple physical pendulum with arbitrary situated barriers imposed on the position of the system. The Aizerman-Gantmakher theory, handling with perturbed solution in points of discontinuity, is used to extend classical method for computing Lyapunov exponents for the multi-degree of freedom mechanical system with rigid barriers imposed on its position. Some examples of identification of attractors in the system of triple pendulum with horizontal barrier are presented, including periodic, quasi-periodic and chaotic attractors. We have focused on the calculation of Lyapunov exponents, however the same methods can be used in the stability and bifurcation analysis of periodic solutions. Let us also note, that the mentioned above methods are suitable for analysis of classical bifurcations occurring in non-smooth systems. For non-classical bifurcations (like grazing bifurcation as an example) the developed methods may not be sufficient.

Acknowledgments The work has been supported by the Ministry of Science and Higher Education under the grant no. 0040/B/T02/2010/38. J. Awrejcewicz acknowledges support of the Alexander von Humboldt Award.

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