# Load-Transfer from an Elastic Fibre to Isotropic Half-Space with Coating

**Igor V. Andrianov, Jan Awrejcewicz, and Dieter Weichert**

## 1 Introduction

We study the problem of load-transfer from fiber inclusion to matrix. Many papers are devoted to the infinite fibre in an elastic space. 3D analog of Melan problem is analysed by Muki and Sternberg [1]. They regard the original fibre as made of two superimposed elastic fibres, the first with the same characteristics as the matrix and treated in the framework of 3D elasticity, the latter with the elastic coefficient equal to difference between those of the actual fibre and of the matrix considered as a 1D continuum. The governing integral equation is obtained by imposing the same average axial strain in the two fictitious bars.

Many researches used as asymptotic parameters ratios $\lambda_1 = R/L, \lambda_2 = E/E_1$ or $\lambda_3 = \frac{E_1}{E}(\frac{R}{L})^2 \ln(\frac{2L}{R})$, where $E, E_1$ are the Young modulus of matrix and fibre, respectively; and $R, L$ are the radius and length of the circular fibre, respectively.

Freund [2] studied a model describing sliding of circular cylindrical fibre along a hole in an elastic solid, and obtained asymptotic solutions for the cases when the fibre is very stiff or very weak in comparison with the matrix material ($\lambda_2 << 1$ and $\lambda_2 >> 1$, respectively). Eshelby [3] and Argatov and Nazarov [4] used parameters $\lambda_1 << 1$ and $\lambda_2 << 1$ and matched asymptotics procedure. Phan-Thien and Kim [6] used parameter $\lambda_3$. If $\lambda_3 >> 1$, then the interfacial shear stress remains almost constant, for $\lambda_3 << 1$ the load transfer occurs over a finite neighbourhood of the fibre end which is near to the free surface and the interfacial shear stress varies as $1/z^3$, where z is the distance from the free surface.

---

I.V. Andrianov and D. Weichert
Department of General Mechanics, TU University of Aachen, Germany, e-mail: igor_andrianov@hotmail.com, weichert@iam.rwth-aachen.de

J. Awrejcewicz
Department of Automatics and Biomechanics, Technical University of Lodz, Poland, e-mail: awrejcew@p.lodz.pl

J. Awrejcewicz (ed.) *Modeling, Simulation and Control of Nonlinear Engineering Dynamical Systems.*

Many papers are also devoted to the problem of load-transfer from a single fibre to the half-space. Keer and Luk [7] formulated a problem of load-transfer by means of Hankel transforms and reduced it to a system of coupled singular integral equations, where the unknown quantities are the normal and the shear stresses acting on the entire surface of the fibre.

In paper by McCartney [9] the equilibrium equations, the interface conditions and other boundary conditions involving stresses are exactly satisfied. Furthermore, two of the four stress-strain relations are satisfied exactly, whereas the remaining two are satisfied in an average sense. Displacement boundary conditions are also satisfied in an average sense. The approach proposed by Rajapakse and Wang [10] is based on the study of interaction between the 1D elastic fibre and the 3D elastic half-space with a cylindrical cavity. The displacement compatibility is achieved along the contact surface between the fibre and the half-space. A variational technique coupled with a boundary integral equation scheme base on a set of exact Green's functions is used in the analysis. The boundary conditions on the top end of the fibre are incorporated into the variational formulation through a set of Lagrange multipliers. Lee and Mura [11] obtained the numerical solution in the case of finite length fibre embedded in elastic space and in elastic half-space.

Movchan and Willis [12] analyzed case when the fibres are held in place by Coulomb friction. The stress and displacement field in the composite and the length of the slipping region are obtained by solving a model problem for a fibre in an elastic half-space in an ambient stress field generated by all other fibres and the applied loading.

Antipov et al. [13] consider a boundary layer problem for an elastic space containing an infinite cylindrical fibre with a frictional interface. In the region where frictional sliding occurs, the transfer of load across the interface is governed by a Coulomb friction law. Outside the slipping region the fibre and the matrix are perfectly bonded. The problem is reduced to a singular integral equation. Lenci and Menditto [14] obtained solution for dilute and highly concentrated fibre composite with a weak interface in the form of improper integrals.

In this paper we analyse load-transfer from single fibre to half-space through interface, when boundary of half-space is rigidly joined with thin elastic coating.

## 2 Governing Equations

In this section, we will consider the case of a single fibre weakly bonded to a surrounding half-space (Fig. 1). Fibre is loaded by uniformly distributed across its cross-section load $P$. We do not take into account body forces; due to the linearity of the problem it can be done using described approach.

We will consider the fibre as 1D continuum without transversal deformation and we will suppose perfect adherence in the direction orthogonal to the fibre-matrix interface. First approximation is based on the inequality $\lambda_2 << 1$. The matrix material is assumed to be isotropic and linear elastic, with elastic constants $E$ and $\nu$. The

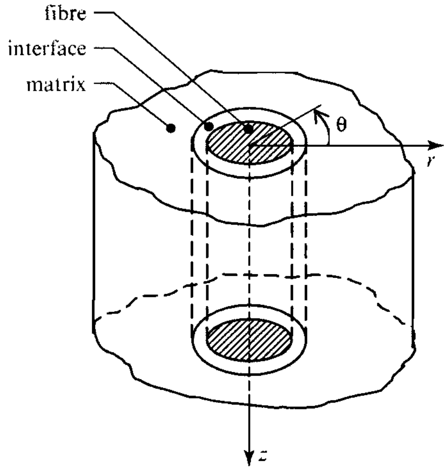


**Fig. 1** The dilute concentration problem: a single fibre embedded in an elastic half-space

axial Young modulus of the circular fibre with radius R and the interface stiffness are denoted by $E_1$ and $k$, respectively. We will use circular cylindrical coordinate system $(r,\theta,z)$; axis of the fibre coincides with the z-axis. The problem is axially symmetric; the axial displacement of the fibre is denoted by $U_f(z)$ and the radial and longitudinal displacement of the matrix by $U_r(r,z)$ and $U_z(r,z)$, respectively. We also denote the interfacial stress by $\tau(z)$ and stresses in the matrix by $\sigma_r(r,z), \sigma_z(r,z), \sigma_\theta(r,z), \tau_{rz}(r,z)$; in our case $\tau(z) = \tau_{rz}(R,z)$.

The interface between fibre and matrix can play an important role in determining the properties of the composite material. Usually, stresses are continuous across the interface, while the displacements may be continuous or discontinuous. In the former case, the interface is called "strong", whereas in the latter case, it is called "weak". We deal with a weak interface described by the spring-layer model which assumes that the interfacial stress is a function of the gap in the displacements. Asymptotic justifications of spring-layer model were obtained by many authors; for example, see [15] and references cited therein. We suppose the material of the interface to be incompressible so Poisson's coefficient of interface is equal to $1/2$. In this case the interface guarantees perfect bonding in normal direction and only tangential sliding is possible [14, 15]:

$$\tau(z) = k(U_f(z) - U_z(R,z)). \tag{1}$$

The parameter $k$ summarizes the mechanical characteristics of the interface and can be computed from the elastic moduli of the interface [15]. For the case of an incompressible interface one has $k = E_i/(3d)$, where $E_i$ is the Yung modus of the interface and $d$ is the thickness of the interface.

We also suppose that fibre is absolutely rigid in radial direction [14]:

$$U_r(r,z) = 0. \tag{2}$$

Displacements and stresses in the matrix can be expressed in terms of the Love potential $\Phi(r,z)$ as follows:

$$U_r(r,z) = -\frac{\partial^2\Phi}{\partial r\partial z}, U_z(r,z) = 2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}, \tag{3}$$

$$\sigma_r(r,z) = 2G\frac{\partial}{\partial z}\left(\nu\nabla^2\Phi - \frac{\partial^2\Phi}{\partial r^2}\right), \tag{4}$$

$$\begin{aligned} \sigma_z(r,z) &= 2G\frac{\partial}{\partial z}\left((2-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right), \\ \tau_{rz}(r,z) &= 2G\frac{\partial}{\partial r}\left((1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right), \end{aligned} \tag{5}$$

where: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, G = \frac{E}{2(1+\nu)}$.

In the absence of body force, the function $\Phi(r,z)$ is biharmonic:

$$\nabla^2\nabla^2\Phi(r,z) = 0. \tag{6}$$

Now let us suppose that matrix is coated by thin elastic layer with the small thickness H, rigidly bonded to the elastic half-space. This model is valid for polymer material with a metal coating [16]. The coating material is assumed to be isotropic and linear elastic, with elastic constants $E_2$ and $\nu_1$.

Due to the small thickness of coating we can treat it as a plate. Then boundary conditions for z = 0 can be written as follows:

$$\frac{E_2 H}{1-\nu_1^2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\frac{\partial^2\Phi}{\partial r\partial z} = -2G\frac{\partial}{\partial z}\left(\nu\nabla^2\Phi - \frac{\partial^2\Phi}{\partial r^2}\right), \tag{7}$$

$$\begin{aligned} \frac{E_2 H^3}{12\left(1-\nu_1^2\right) r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left[2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right]\right)\right)\right) &= \\ = 2G\frac{\partial}{\partial z}\left((2-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right). \end{aligned} \tag{8}$$

and

$$U_r, U_z, \sigma_r, \sigma_z, \sigma_{\theta\theta}, \tau_{rz} \to 0 \text{ for } z \to \infty. \tag{9}$$

## 3 Asymptotic Simplification of Boundary Conditions

Let us introduce nondimensional variables $r_1 = r/R, \xi = z/R$. Then boundary conditions (7), (8) for $\xi = 0$ can be rewritten as follows:

$$d_1\left(\frac{\partial^2}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial}{\partial r_1} - \frac{1}{r_1^2}\right)\frac{\partial^2\Phi}{\partial r_1\partial\xi} = -\frac{\partial}{\partial\xi}\left(\nu\nabla^2\Phi - \frac{\partial^2\Phi}{\partial r_1^2}\right) \tag{10}$$

$$\frac{d_2}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial}{\partial r_1}\left(\frac{1}{r_1}\frac{\partial}{\partial r_1}\left(r_1\frac{\partial}{\partial r_1}\left[2(1-\nu)\nabla_1^2\Phi-\frac{\partial^2\Phi}{\partial\xi^2}\right]\right)\right)\right) \tag{11}$$

$$=\frac{\partial}{\partial\xi}\left((2-\nu)\nabla_1^2\Phi-\frac{\partial^2\Phi}{\partial\xi^2}\right), \tag{12}$$

where:

$$\nabla_1^2=\frac{\partial^2}{\partial r_1^2}+\frac{1}{r_1}\frac{\partial}{\partial r_1}+\frac{\partial^2}{\partial\xi^2}d_1=\frac{E_2H}{2G(1-\nu_1^2)R},d_2=\frac{E_2H^3}{24G(1-\nu_1^2)R^3}.$$

Thin coating cannot influence sufficiently the normal stresses. On the other hand coating layer has a large rigidity in the tangential direction that is why one can suppose radial displacements on the boundaries equal zero [5]. Taking into account these assumptions, one can simplify boundary conditions (10), (12). In the first approximation one has (for $\xi=0$):

$$\left(\frac{\partial^2}{\partial r_1^2}+\frac{1}{r_1}\frac{\partial}{\partial r_1}-\frac{1}{r_1^2}\right)\frac{\partial^2\Phi}{\partial r_1\partial\xi}=0, \tag{13}$$

$$\frac{\partial}{\partial\xi}\left[(2-\nu)\nabla_1^2\Phi-\frac{\partial^2\Phi}{\partial\xi^2}\right]=0. \tag{14}$$

From the physical standpoint one has in this case an inextensible membrane ideally bonded to the matrix at the half-space boundary.

In the original variables boundary conditions (13), (14) can be written as follows (for $z=0$):

$$\frac{\partial\Phi}{\partial z}=0, \tag{15}$$

$$\frac{\partial^3\Phi}{\partial z^3}=0. \tag{16}$$

## 4 A Single Fibre Embedded in the Half-Space

Let us use for solving boundary value problem (6), (15), (16) the cosine Fourier transform

$$\overline{\Phi}(r,s)=\int_0^\infty\Phi(r,z)\cos(sz)dz. \tag{17}$$

Partial differential equation (6) is transformed to the ordinary differential equation

$$\nabla_2^2\nabla_2^2\overline{\Phi}(r,s)=0, \tag{18}$$

where: $\nabla_2^2=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-s^2$.

A general solution to the ordinary differential equation (2) is as follows

$$\overline{\Phi}(r,s) = AK_0(sr) + BsrK_1(sr) + CI_0(sr) + DsrI_1(sr), \tag{19}$$

where $K_0, K_1, I_0$ and $I_1$ are the modified Bessel functions [8].

From conditions (9) one obtains $C = D = 0$, while condition (2) yields

$$A = -\frac{sRK_0(sR)}{K_1(sR)}B. \tag{20}$$

Then from (1), (5), (19) and (20) one obtains

$$\bar{\tau}(s) = \bar{\tau}_{rz}(R,s) = 4BG(1-\nu)s^3K_1(sR), \tag{21}$$

$$\bar{U}_z(R,s) = \frac{R}{G}\bar{\tau}(s)g(sR), \tag{22}$$

where:

$$g(sR) = -\frac{1}{sR}\frac{K_0(sR)}{K_1(sR)} - \frac{1}{4(1-\nu)}\left(\frac{K_0(sR)}{K_1(sR)}\right)^2 + \frac{1}{4(1-\nu)}. \tag{23}$$

In what follows it will be useful to obtain asymptotics of function $g(sR)$ for $s \to 0$ and $s \to \infty$. Using formulas (9.6.8), (9.6.9), and (9.7.2) from [8] one gets

$$g(sR) \sim \ln(sR) + a, \text{ for s } \to 0, \tag{24}$$

$$g(sR) \sim -\frac{3-4\nu}{4(1-\nu)sR}, \text{ for s } \to \infty, \tag{25}$$

where: $a = \gamma - \ln 2 + \frac{1}{4(1-\nu)} > 0, \gamma = 0.577215649\ldots$ (the Euler constant).

Fibre equilibrium condition can be written as follows [14]:

$$E_1\frac{d^2U_f}{dz^2} + \frac{2}{R}\tau(z) = 0. \tag{26}$$

Boundary conditions associated with Eq. (26) have the following form

$$E_1 = \frac{dU_f}{dz} = P, \text{ for } z = 0, \tag{27}$$

$$U_f \to 0, \text{ for } z \to \infty. \tag{28}$$

Application of cosine Fourier transform (17) to the boundary value problem (26)–(28) yields

$$-s^2\bar{U}_f(s) + \frac{2}{E_1R}\bar{\tau}(s) + \frac{P}{E_1} = 0. \tag{29}$$

From condition (1) and relations (22), (23) and (29) we find $\bar{\tau}(s), \bar{U}_z(R,s)$ and $\bar{U}_f(s)$. Using then inverse cosine Fourier transform (26) of the form

$$U_z(R,z)=\frac{2}{\pi}\int\limits_0^\infty \bar{U}_z(R,s)\cos(sz)ds, \tag{30}$$

one obtains

$$\tau(\xi)=-\frac{P}{R}\int\limits_0^\infty M(\varphi)\cos(\varphi\xi)d\varphi, \tag{31}$$

$$U_z(1,\xi)=-\frac{2(1+\nu)PR}{\pi E_1}\int\limits_0^\infty M(\varphi)g(\varphi)\cos(\varphi\xi)d\varphi, \tag{32}$$

$$U_f(\xi)=-\frac{PR}{\pi E_1}\int\limits_0^\infty\left[\frac{E_1}{kR}-\frac{E_1}{G}g(\varphi)\right]M(\varphi)\cos(\varphi\xi)d\varphi, \tag{33}$$

$$\sigma_r(1,\xi)=-\frac{P}{\pi}\int\limits_0^\infty\left[2\varphi g(\varphi)+\frac{K_0(\varphi)}{K_1(\varphi)}\right]M(\varphi)\sin(\varphi\xi)d\varphi, \tag{34}$$

$$\sigma_z(1,\xi)=\frac{P}{\pi}\int\limits_0^\infty\left[2\varphi g(\varphi)-\frac{\nu}{1-\nu}\frac{K_0(\varphi)}{K_1(\varphi)}\right]M(\varphi)\sin(\varphi\xi)d\varphi, \tag{35}$$

$$\varphi=sR,\ k_1^2=\frac{kR}{E_1},\ M(f)=\frac{2}{\frac{f^2}{k_1^2}-\frac{E_1f^2g(f)}{G}+2}. \tag{36}$$

Formulae (31)–(35) differ from formulae for the problem of single fibre embedded in the space obtained in [14] only by factor 2.

Now we will estimate integrals (31)–(35). First of all, we rewrite them in the following form:

$$\tau(\xi)=\frac{P}{R}I_1, \tag{37}$$

$$U_z(1,\xi)=\frac{2(1+\nu)PR}{\pi E_1}I_2, \tag{38}$$

$$U_f(\xi)=-\frac{P}{\pi k}I_1+\frac{PR}{\pi G}I_2, \tag{39}$$

$$\sigma_r(1,\xi)=-\frac{P}{\pi}(2I_3+I_4), \tag{40}$$

$$\sigma_z(1,\xi)=-\frac{P}{\pi}\left(2I_3-\frac{\nu}{1-\nu}I_4\right). \tag{41}$$

Asymptotics of function $M(\varphi)$ are of the following form

$$M(\varphi)\to 1,\ \text{for}\ \varphi\to 0, \tag{42}$$

$$M(\varphi) \to \frac{2k_1^2}{\varphi^2}, \text{ for } \varphi \to \infty. \tag{43}$$

Asymptotic expressions (42), (43) give a possibility to obtain the following interpolation function (valid for all values of $\varphi$):

$$M(\varphi) \approx \frac{2k_1^2}{\varphi^2 + 2k_1^2}, \tag{44}$$

$$I_1 \approx -\frac{\pi P k_1}{\sqrt{2}R} \exp\left(-\sqrt{2}k_1\xi\right), \tag{45}$$

and, respectively, one has

$$\tau(\xi) \approx -\frac{\pi P k_1}{\sqrt{2}R} \exp\left(-\sqrt{2}k_1\xi\right), \tag{46}$$

$$\tau_{\max}(k) \approx \frac{\pi P k_1}{\sqrt{2}R}. \tag{47}$$

We compared this value of $\tau_{max}$ with numerical data (see [6]). Discrepancy between approximate analytical and numerical results is not sufficient.

Now we will analyse integral $I_2$. Asymptotic expressions for function $g(\varphi)M(\varphi)$ are as follows

$$g(\varphi)M(\varphi) \to \ln\varphi + a, \text{ for } \varphi \to 0, \tag{48}$$

$$g(\varphi)M(\varphi) \to -\frac{2a_1k_1^2}{\varphi^3}, \text{ for } \varphi \to \infty, \tag{49}$$

where: $a_1 = \frac{(3-4\nu)a}{R}$.

Let us suppose integral $I_2$ as follows:

$$I_2 = I_2^{(1)} + I_2^{(2)}, \tag{50}$$

where: $I_2^{(1)} = I_2 - I_2^{(2)}$, $\quad I_2^{(2)} = \int\limits_0^\infty f(\varphi)\cos(\varphi\xi)d\varphi$, $\quad f(\varphi) = \begin{cases} \ln\varphi, & 0 < \varphi < 1, \\ 0, & 1 \le \varphi. \end{cases}$

Calculate integral $I_2^{(2)}$, one obtains:

$$I_2^{(2)}(\xi) = -\frac{Si(\xi)}{\xi}, \tag{51}$$

where: $Si(\xi)$ is familiar sine integral [8].

Expression under the integral sign $M_2(\varphi)$ in the integral $I_2^{(1)}$ has the following asymptotics:

$$M_2(\varphi) \to a, \text{ for } \varphi \to 0, \tag{52}$$

$$M_2(\varphi) \to -\frac{2a_1k_1^2}{\varphi^3}, \text{ for } \varphi \to \infty. \tag{53}$$

It means that for all values $\varphi$ one can use the following interpolation function for $M_2(\varphi)$:

$$M_2(\varphi) \approx \frac{2k_1^2 a(1-a_2\varphi)}{(1+\varphi^2)(2k_1^2+\varphi^2)}, \tag{54}$$

where: $a_2 = a_1/a$.

Using residual theorem, one obtains:

$$I_2^{(1)} = \frac{\sqrt{2}ak_1}{2k_1^2-1}[\sqrt{2}k_1 \exp(-\xi) - \exp(-\sqrt{2}k_1^{\xi})]. \tag{55}$$

Now we will analyse integral $I_3$. Let us obtain asymptotic expressions for function $\varphi g(\varphi)M(\varphi)$:

$$\varphi g(\varphi)M(\varphi) \to \varphi(\ln\varphi + a), \text{ for } \varphi \to 0, \tag{56}$$

$$\varphi g(\varphi)M(\varphi) \to -\frac{2a_1k_1^2}{\varphi^2}, \text{ for } \varphi \to \infty. \tag{57}$$

Let us divide integral $I_3$ into two following parts

$$I_3 = I_3^{(1)} + I_3^{(2)}, \tag{58}$$

where: $I_3^{(1)} = I_3 - I_3^{(2)}$, $I_3^{(2)} = \int\limits_0^\infty e^{-\varphi}\varphi \ln\varphi \sin(\varphi\xi)d\varphi$.

Computation of integral $I_3^{(2)}$ yields

$$I_3^{(2)} = \frac{2}{(1+\xi^2)^2}\left[2(1-\gamma)\xi - \xi\ln(1+\xi^2) + \arctan\xi(1-\xi^2)\right]. \tag{59}$$

Expression under the integral $I_3^{(1)}$ has the following asymptotics

$$M_3(\varphi) \to a\varphi, \text{ for } \varphi \to 0, \tag{60}$$

$$M_3(\varphi) \to -\frac{2a_1k_1^2}{\varphi^2}, \text{ for } \varphi \to \infty. \tag{61}$$

Using asymptotics (60), (61) one can construct interpolation function for $M_3(\varphi)$, valid for all values of $\varphi$ of the form

$$M_3(\varphi) \approx 2k_1^2 a \frac{\varphi(1-a_2\varphi)}{(1+\varphi^2)(2k_1^2+\varphi^2)}. \tag{62}$$

On the other hand the residual theorem, yields:

$$I_3^{(1)} = \frac{2ak_1^2}{2k_1^2-1}[\exp(-\xi) - \exp(-\sqrt{2}k_1\xi)].$$

Finally let us analyse integral $I_4$. Function $\frac{k_0(\varphi)}{k_1(\varphi)}M(\varphi)$ has the following asymptotics:

$$\frac{k_0(\varphi)}{k_1(\varphi)}M(\varphi) \to -\varphi(\ln\varphi + a), \text{ for } \varphi \to 0, \tag{63}$$

$$\frac{k_0(\varphi)}{k_1(\varphi)}M(\varphi) \to \frac{2k_1^2}{\varphi^2}, \text{ for } \varphi \to \infty. \tag{64}$$

In what follows we assume the following form of integral $I_4$

$$I_4 = I_4^{(1)} + I_4^{(2)}, \quad I_4^{(1)} = I_4 - I_4^{(2)}. \tag{65}$$

Expression under the integral sign $M_4(\varphi)$ in the integral $I_4^{(1)}$ exhibits the following asymptotics:

$$M_4(\varphi) \to -a\varphi \text{ for } \varphi \to 0, \tag{66}$$

$$M_4(\varphi) \to \frac{2k_1^2}{\varphi^2} \text{ for } \varphi \to \infty. \tag{67}$$

Interpolation functions valid for all values of $\varphi$ can be written as follows

$$M_4(\varphi) \approx -2k_1^2 a \frac{\varphi(1-\varphi/a)}{(1+\varphi^2)(2k_1^2+\varphi^2)}. \tag{68}$$

Therefore, the residual theorem yields the following result

$$I_4^{(1)} = -\frac{2ak_1^2}{2k_1^2 - 1}[\exp(-\xi) - \exp(-\sqrt{2}k_1^{\xi})]. \tag{69}$$

## 5 Conclusions

The obtained results can be used for investigation of a composite fracture. Solved problems in the field of Civil Engineering model, the behaviour of piles or piers embedded in soil media, which exhibit a linear elastic response in the working-load range. Analytic solutions presented in this paper will be useful in evaluating test results calculated by boundary elements and finite element methods.

**Acknowledgements** This work was supported by the German Research Foundation (Deutsche Forschungs-gemeinschaft), grant No. WE 736/25-1 (for I.V. Andrianov and D. Weichert).

## References

1. Muki R, Sternberg E (1969) On the diffusion of an axial load from an infinite cylindrical bar embedded in an elastic medium, International Journal of Solids and Structures 5(6), 587–605.
2. Freund LB (1992) The axial force needed to slide a circular fibre along a hcle in an elastic material and implications for fibre pull-out, European Journal of Mechanics A-Solids 11(1), 1–19.

3. Eshelby JD (1982) The stresses on and in a thin inextensible fibre in a stretched elastic medium, Engineering Fracture Mechanics 16(3), 453–455.
4. Argatov II, Nazarov SA (1996) Asymptotic analysis of problems on junctions of domains of different limit dimensions. A body pierced by a thin rod, Izvestiya Mathematics 60(1), 1–37.
5. Phan-Thien N, Kim S (1994) Microstructures in Elastic Media: Principles and Computational Methods, Oxford University Press, New York.
6. Keer LM, Luk VK (1979) Stress analysis for an elastic half space containing an axially-loaded rigid cylindrical rod, International Journal of Solids and Structures 15, 605–627.
7. McCartney LN (1989) New theoretical model of stress transfer between fibre and matrix in a unaxialy fibre-reinforced composite, Proceedings of the Royal Society of London A 425, 215–244.
8. Rajapakse RKND, Wang Y (1990) Load-transfer problem for transversely isotropic elastic media, Journal of Engineering Mechanics 116(12), 2643–2662.
9. Ven-Gen Lee, Mura T (1994) Load transfer from a finite cylindrical fiber into an elastic half-space, Journal of Applied Mechanics 61, 971–975.
10. Movchan AB, Willis JR (1997) Asymptotic analysis of reinforcement by frictional fibres, Proceedings of the Royal Society of London A 453, 757–784.
11. Antipov YA, Movchan AB, Movchan NV (2000) Frictional contact of fibre and an elastic solid, Journal of Mechanics and Physics of Solids 48, 1413–1439.
12. Lenci S, Menditto G (2000) Weak interface in long fiber composites, International Journal of Solids and Structures 37, 4239–4260.
13. Geymonat G, Krasucki F, Lenci S (1999) Mathematical analysis of a bonded joint with a soft thin adhesive, Mathematics and Mechanics of Solids 4, 201–225.
14. Alexandrov VM, Mkhitaryan SM (1983) Contact Problems for Bodies with Thin Coatings and Inclusions, Nauka, Moscow.
15. Andrianov IV, Danishevs'kyy VV (2007) Load-transfer to an orthotropic fibre-reinforced composite strip via an elastic element, Technische Mechanik 27(1), 28–36.
16. Abramowitz M, Stegun IA (eds.) (1965) Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables, Dover Publications, New York.