Computational Methods in Engineering and Science, Iu et al. (eds)
© 2003 Swets & Zeitlinger, Lisse, ISBN 90 5809 567 3

Analysis of complex parametric vibrations of plates and shells using the Bubnov-Galerkin approach

J. Awrejcewicz
Technical University of Lodz, Department of Automatics and Biomechanics, Poland

A.V. Krys'ko
Saratov State University, Department of Mathematics, Saratov, Russia

ABSTRACT: The Bubnov-Galerkin method is applied to reduce partial differential equations governing flexible plates and shells dynamics to a system with finite degrees-of-freedom. Chaotic behaviour of systems with various degrees-of-freedom is analyzed.

1 MULTIBODY DYNAMICAL SYSTEMS

Chaotic vibrations exhibited by lumped systems with many degrees of freedom are quite rarely investigated. However, recently remarkable progress in this field has been observed: hydrodynamic processes governed by ordinary differential equations have been investigated (Swinney & Gollub 1981), the finite dimensional discretized (with respect to spatial coordinates) models of Ginzburg-Landau equations (L'vov et al. 1981), multidimensional models of radiophysical systems governing the dynamics of coupled oscillators and generators (Waller & Kapral 1984), as well as chains of oscillators and generators have been analysed. In the majority of the eited works a problem of modeling a continuous system by a lumped (discrete) system governed by ordinary differential equations is addressed.

Nowadays various approximational methods are applied to construct lumped systems. In this work we use the Bubnov-Galerkin method, whieh bas been successfully applied to different types of differential equations: elliptie, hyperbolic and parabolic ones. In the monographs (Krysko & Kutsemako 1999, Awrejcewicz & Krysko 2003) a review of the Bubnov-Galerkin method (MBG) is given ineluding a discussion of its convergence for various classes of differential equations.

2 PROBLEM FORMULATION AND THE BUBNOV-GALERKIN METHOD

The equations governing dynamics of a reetangular shell including both transversal and longitudinal harmonic excitations have the following form (Krysko & Kutsemako 1999, Awrejcewicz & Krysko 2003):

$$\frac{\partial^2 w}{\partial t} + \varepsilon \frac{\partial w}{\partial t} + \frac{1}{12(1-\mu^2)}\left[\frac{1}{\lambda^2}\frac{\partial^4 w}{\partial x^4} + \lambda^2 \frac{\partial^4 w}{\partial y^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2}\right] +$$

$$+\left[P_x \frac{\partial^2 w}{\partial x^2} + P_y \frac{\partial^2 w}{\partial y^2}\right] - L(w,F) - \nabla_k^2 F + k_x P_x + k_y P_y - q = 0\,, \qquad (1)$$

$$\frac{1}{\lambda^2}\frac{\partial^4 F}{\partial x^4} + \lambda^2 \frac{\partial^4 F}{\partial y^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \nabla_k^2 w + \frac{1}{2}L(w,w) = 0\,, \qquad (2)$$

where:

$$\nabla_k^2 = k_x \frac{\partial^2}{\partial x^2} + k_y \frac{\partial^2}{\partial y^2}, \qquad L(w,w) = 2\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right],$$

$$L(w,F) = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 F}{\partial x \partial y}. \tag{3}$$

The system of equations (1)–(2) is already in the non-dimensional form, whereas the relations between dimensional and non-dimensional parameters read:

$$w = 2h\overline{w}, \quad x = a\overline{x}, \quad y = b\overline{y}, \quad \lambda = a/b, \quad F = E(2h)^3\overline{F},$$

$$k_x = \frac{2h}{a^2}\overline{k}_x, \quad k_y = \frac{2h}{b^2}\overline{k}_y, \quad q = \frac{E(2h)^4}{a^2b^2}\overline{q},$$

$$t = \frac{ab}{2h}\sqrt{\frac{\rho}{gE}}\,\overline{t}, \quad \varepsilon = \frac{2h}{ab}\sqrt{\frac{gE}{\rho}}\,\overline{\varepsilon}, \quad P_x = \frac{E(2h)^3}{b^2}\overline{P}_x, \quad P_y = \frac{E(2h)^3}{a^2}\overline{P}_y. \tag{4}$$

Let us denote the left hand sides of equations (1)–(2) by ϕ_1 and ϕ_2, respeetively. Hence, the equations have the form:

$$\Phi_1\left(\frac{\partial^2 w}{\partial^2 x}, \frac{\partial^2 F}{\partial^2 x}, k_x, k_y, P_x, P_y, q, t, \ldots\right) = 0, \qquad \Phi_2\left(\frac{\partial^2 w}{\partial^2 x}, \frac{\partial^4 F}{\partial^4 x}, k_x, k_y \ldots\right) = 0. \tag{5}$$

In addition, the eorresponding boundary conditions should be attached. Since the exact solution of the formulated boundary value problem is not known, the MBG method with higher approximations is applied. We assume the following form of the unknown functions

$$w = \sum_{i,j} A_{ij}(t)\varphi_{ij}(x,y), \qquad F = \sum_{i,j} B_{ij}(t)\psi_{ij}(x,y), \tag{6}$$

$$i = 1,2,\ldots,M_x;\ j = 1,2,\ldots,M_y.$$

Applying the MBG procedure to equations (5) and taking into aeeount (6), we get

$$\begin{aligned}
&\sum_{vz}\Big[\sum_{ij}\Big(\frac{\partial^2 A_{ij}}{\partial t^2} + \varepsilon\frac{\partial A_{ij}}{\partial t} - q + k_x P_x + k_y P_y\Big) I_{3,vz\,ij} + \sum_{ij} A_{ij} I_{1,vz\,ij} - \\
&- \sum_{ij} B_{ij} I_{2,vzij} + \sum_{ij} A_{ij} I_{5,vzij} - \sum_{ij} A_{ij} \sum_{kl} B_{kl} I_{4,vzijkl}\Big] = 0, \\
&\sum_{vz}\Big[\sum_{ij} A_{ij} I_{7,vz\,ij} + \sum_{ij} B_{ij} I_{8,vz\,ij} + \sum_{ij} A_{ij} \sum_{kl} A_{kl} I_{6,vz\,ij\,kl}\Big] = 0,
\end{aligned} \tag{7}$$

$$v,i,k = 1,2,\ldots,M_x; \quad z,j,l = 1,2,\ldots,M_y$$

In the above symbol $\sum_{vz}[*]$, standing before each of the equations of the system (7), should be interpreted as the vz system with similar form, and the integrals of the MBG procedure are:

$$I_{1,vzij} = \int_0^1\!\!\int_0^1 \frac{1}{12(1-\mu^2)}\left[\frac{1}{\lambda^2}\frac{\partial^4\varphi_{ij}}{\partial x^4} + \lambda^2\frac{\partial^4\varphi_{ij}}{\partial y^4} + 2\frac{\partial^4\varphi_{ij}}{\partial x^2\partial y^2}\right]\varphi_{vz}\,dxdy,$$

$$I_{2,vzij} = \int_0^1\int_0^1 \left[k_y \frac{\partial^2 \psi_{ij}}{\partial x^2} + k_x \frac{\partial^2 \psi_{ij}}{\partial y^2} \right] \varphi_{vz} dxdy, \qquad I_{3,vzij} = \int_0^1\int_0^1 \varphi_{vz} dxdy,$$

$$I_{4,vzijkl} = \int_0^1\int_0^1 \left[\frac{\partial^2 \varphi_{ij}}{\partial x^2}\frac{\partial^2 \psi_{kl}}{\partial y^2} + \frac{\partial^2 \varphi_{ij}}{\partial y^2}\frac{\partial^2 \psi_{kl}}{\partial x^2} - 2\frac{\partial^2 \varphi_{ij}}{\partial x \partial y}\frac{\partial^2 \psi_{kl}}{\partial x \partial y} \right] \varphi_{vz} dxdy,$$

$$I_{5,vzij} = \int_0^1\int_0^1 \left[\frac{\partial^2 \varphi_{ij}}{\partial x^2} P_x + \frac{\partial^2 \varphi_{ij}}{\partial y^2} P_y \right] \varphi_{vz} dxdy, \tag{8}$$

$$I_{6,vzijkl} = \int_0^1\int_0^1 \left[\frac{\partial^2 \varphi_{ij}}{\partial x^2}\frac{\partial^2 \varphi_{kl}}{\partial y^2} - \frac{\partial^2 \varphi_{ij}}{\partial x \partial y}\frac{\partial^2 \varphi_{kl}}{\partial x \partial y} \right] \psi_{vz} dxdy,$$

$$I_{7,vzij} = \int_0^1\int_0^1 \left[k_y \frac{\partial^2 \varphi_{ij}}{\partial x^2} + k_x \frac{\partial^2 \varphi_{ij}}{\partial y^2} \right] \psi_{vz} dxdy,$$

$$I_{8,vzij} = \int_0^1\int_0^1 \left[\frac{1}{\lambda^2}\frac{\partial^4 \psi_{ij}}{\partial x^4} + \lambda^2 \frac{\partial^4 \psi_{ij}}{\partial y^4} + 2\frac{\partial^4 \psi_{ij}}{\partial x^2 \partial y^2} \right] \psi_{vz} dxdy.$$

The integrals (8) except for (possibly) $I_{3,vzij}$, if the transversal load q is applied not to the whole shell surface, are computed along the whole middle shell surface.

To conclude, the derived system (7) consists of $M_x \times M_y$ second order differential equations with respect to time and of the linear algebraic equations with respeet to B_{ij}.

The initial conditions have the following form

$$w|_{t=0} = w_0, \qquad \left.\frac{\partial w}{\partial t}\right|_{t=0} = 0, \tag{9}$$

where w_0 is either taken from the corresponding statical problem or is defined using another approacb.

Assuming the loading terms, the system of equations (7) is solved using the numerical method, and A_{ij} and B_{ij} are obtained. Next, the found values of A_{ij} and B_{ij} arc substituted into (6), and the being sought funetions w, F are finally found.

3 RESULTS

As an example the squared plate ($\lambda = 1$, $\varepsilon = 1$), supported hy balls on its contour on flexible non-stretched (non-compressed) ribs and excited longitudinally by $P_x = P_{x_0}(1 - \sin\omega_2 t)$, is investigated. The computations are carried out for fixed ω_2 with variations of the control parameter $P_{x_0} = 1 \pm 18$. The numerical results are used to construct the dependencies $w_{\max}(P_{x_0})$, $w_{ij}(\dot{w}_{ij})$, power spectrum, the Poincaré sections and the Lyapunov exponents.

Some of the mentioned characteristics are shown in Figures 1–4. In Figure 1 the dependencc of maximal deflection in the center of squared plate, versus the longitudinal load P_{x_0}, is reported. The curve 1 is obtained using a first order approximation (i.e. in the relation (6) $\varphi_{ij} = \sin(i\pi x)\sin(j\pi y)$, $\Psi_{ij} = \sin(i\pi x)\sin(j\pi y)$, and $i = j = 1$); the curve 2 is obtained using the 9th order approximation ($i = j = 3$); the curve 3 is obtained using the 25th order approximation ($i = j = 5$); and finally, the curve 4 is obtained using 49th order approximation ($i = j = 7$). The derived results are divided into four intervals: $I - 1 \leqslant P_{x_0} \leqslant 4.5$, $II - 4.5 \leqslant P_{x_0} \leqslant 5.5$, $III - 5.5 \leqslant P_{x_0} \leqslant 7.5$, $IV - 7.5 \leqslant P_{x_0} \leqslant 18.5$.

The intervals $0 \leqslant P_{x_0} \leqslant 1$ and II correspond to intervals of stablc equilibrium. In the interval I–II, for all approximations (practically) thc same rcsults are obtained for all earlier mentioned charaeteristies

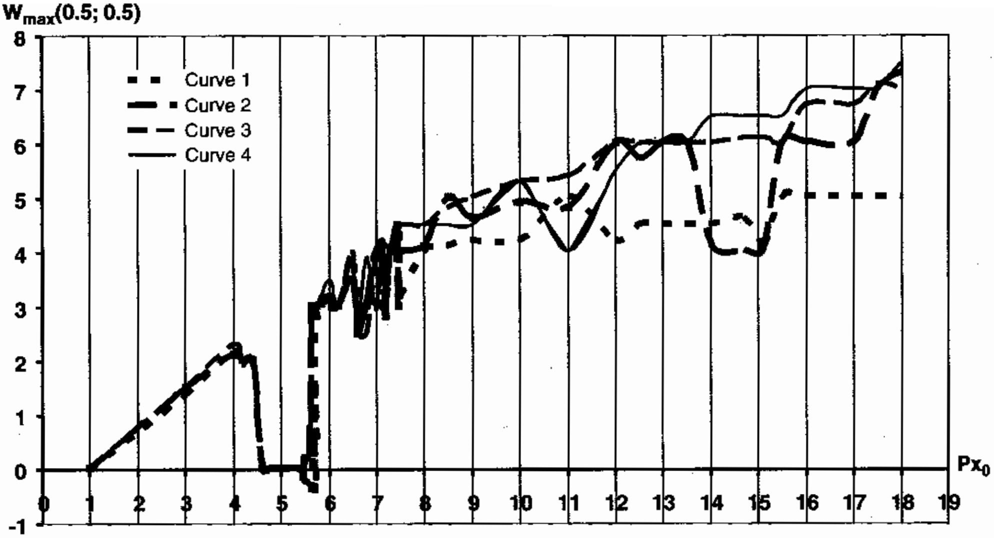

Figure 1. The maximal deflection w_{max} (0.5; 0.5) versus the longitudinal load P_{x_0}.

(the dependencies $(w_{ij}(t))$, $(\dot{w}_{ij}(t))$, power spectra, Poincaré sections and Lyapunov exponents). In the interval *III*, in practice a convergence is achieved when forty nine series terms are used. The dependencies $\sum_{ij}^{M_x=M_y} A_{ij}(t)$; $\sum_{ij}^{y} A_{ij}[\sum_{ij}\dot{A}_{ij}]$ and power spectra are reported in Figures 2–4 (the assoeiated parameters are attaehed to the figures). Analysis of the data in Figure 2 for $P_{x_0} = 5.65$ shows that the results obtained using the first approximation are qualitatively different from the results obtained for $M_x = M_y = 3; 5; 7$.

4 CONCLUSIONS

All of the characteristics for $M_x = M_y = 3; 5; 7$ practically overlap; the vibrations are quasi-periodic and (for higher approximations) a strange chaotic attractor is detected, whieh eonsists of intervals of fast and slow time seales. The relaxation character of vihrations is typical for all modes and has high frequency inclusions on the top of impulses. In zones *I–IV* the same results are obtained for all approximations. In the first approximation (one-degree-of-freedom) the solutions overlap with higher approximations in zones *I–III*. Beginning with zone *IV*, the solutions in the first approximation are not useful to approximate the vibration process and they are qualitatively different from higher order approximations. The reported dependencies obtained for $M_x = M_y = 3; 5; 7$ in practice are the same for both $\sum_{ij} A_{ij}(t)$, as well as for each term of the series $A_{ij}(t)$, phase portraits and power spectra. Increasing the parameter P_{x_0} the system begins to lose regular vibrations in the vicinity of region *V*, and to "forget" its initial state, and it transits into a zone of chaotic vibrations. All of approximations, in spite of the first one, characterize the chaotic vibrations. In addition, zones where neutral curvc of vibrations change appear in "stiff" manner, i. e. series of "stiff" stability loss is observed in the vibration process (see Fig. 3). Note that the results using approximation of 25 and 49 terms are similar. Recall that in the Lorenz model the most sensitive parameter is that associated with modes number. In our model governed by von Kármán equations, this property is not detected. On the contrary, the system behaves similarly for all modes for $P_{x_0} \in [0.5, 5]$. For $P_{x_0} \in [5.5; 7.5]$ higher approximations also converge to one solution, and for $P_{x_0} > 7.5$ higher approximations describe a chaotic plate dynamics. Therefore, here a coupling scheme is observed. The investigation of the characteristics reported shows that each term of the series A_{ij}, for fixed P_{x_0} values, fully describes the character of vibrations (a synchronization of subsystems is ohserved). One may also conclude, that beginning from $P_{x_0} > 7.5$ the so called "multimode" turbulence (or "true" turbulence) is observed.

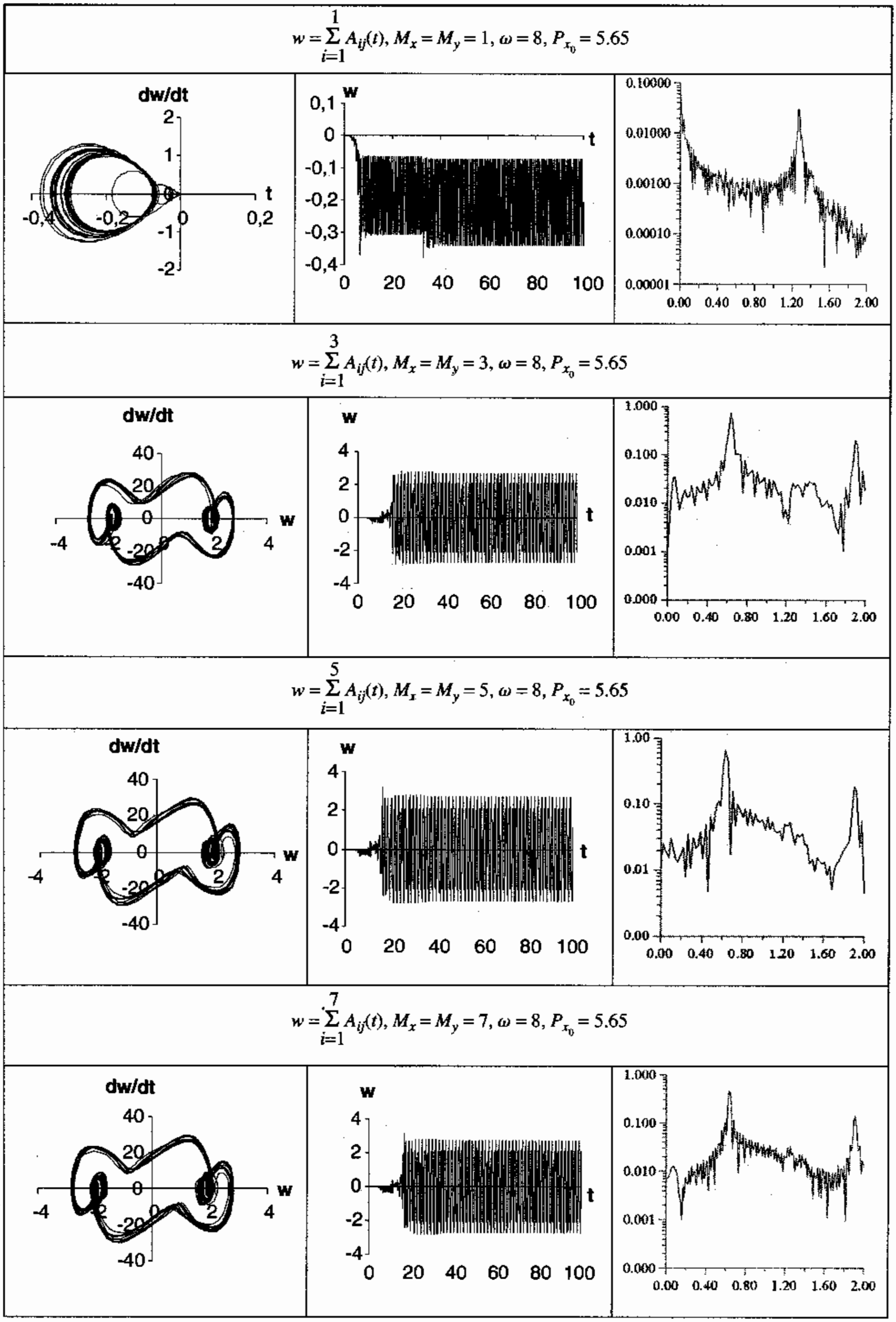

Figure 2. Phase portraits, time histories and power speetra for the attaehed parameters.

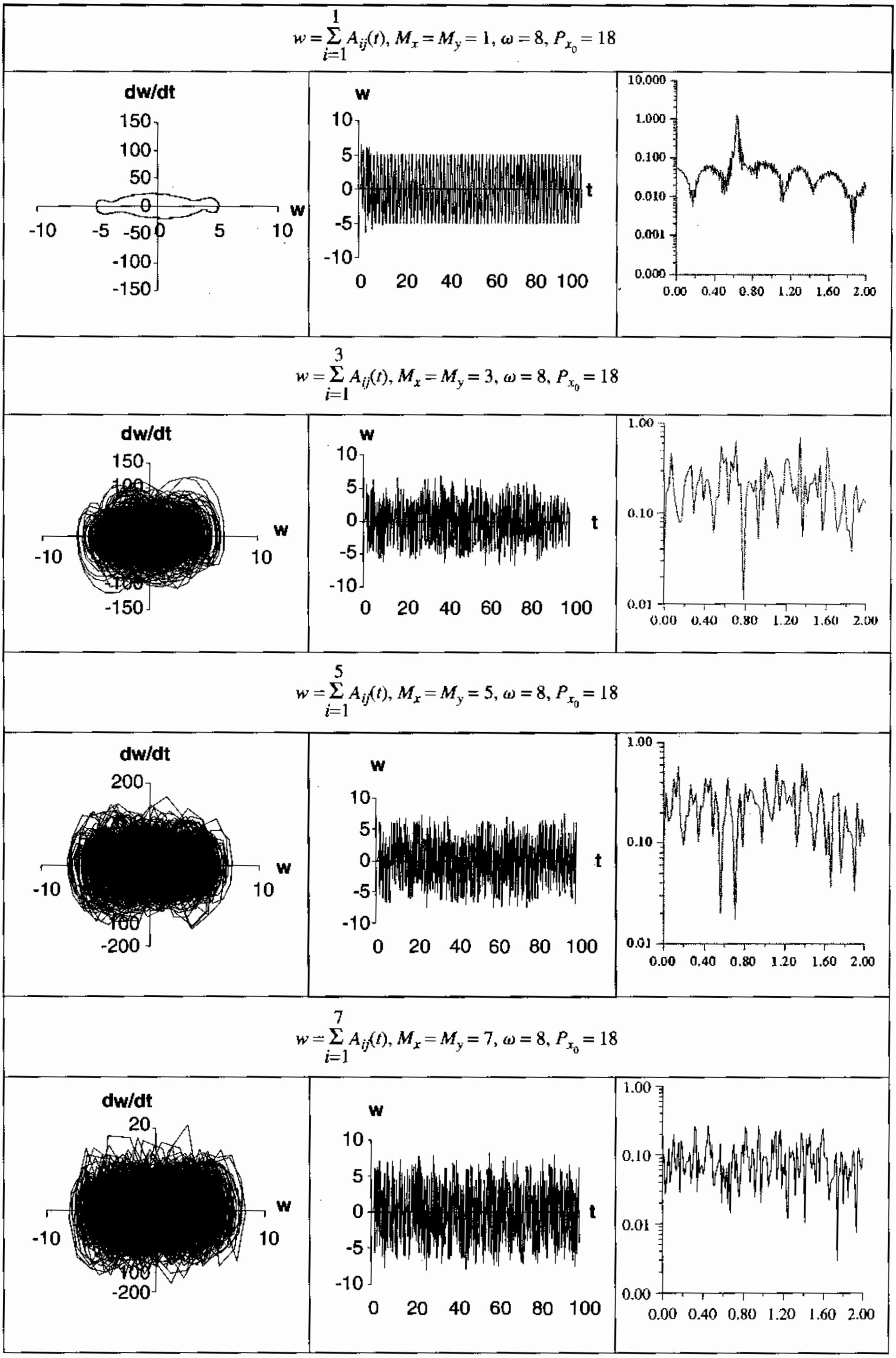

Figure 3. Phase portraits, time histories and power spectra for the attached parameters.

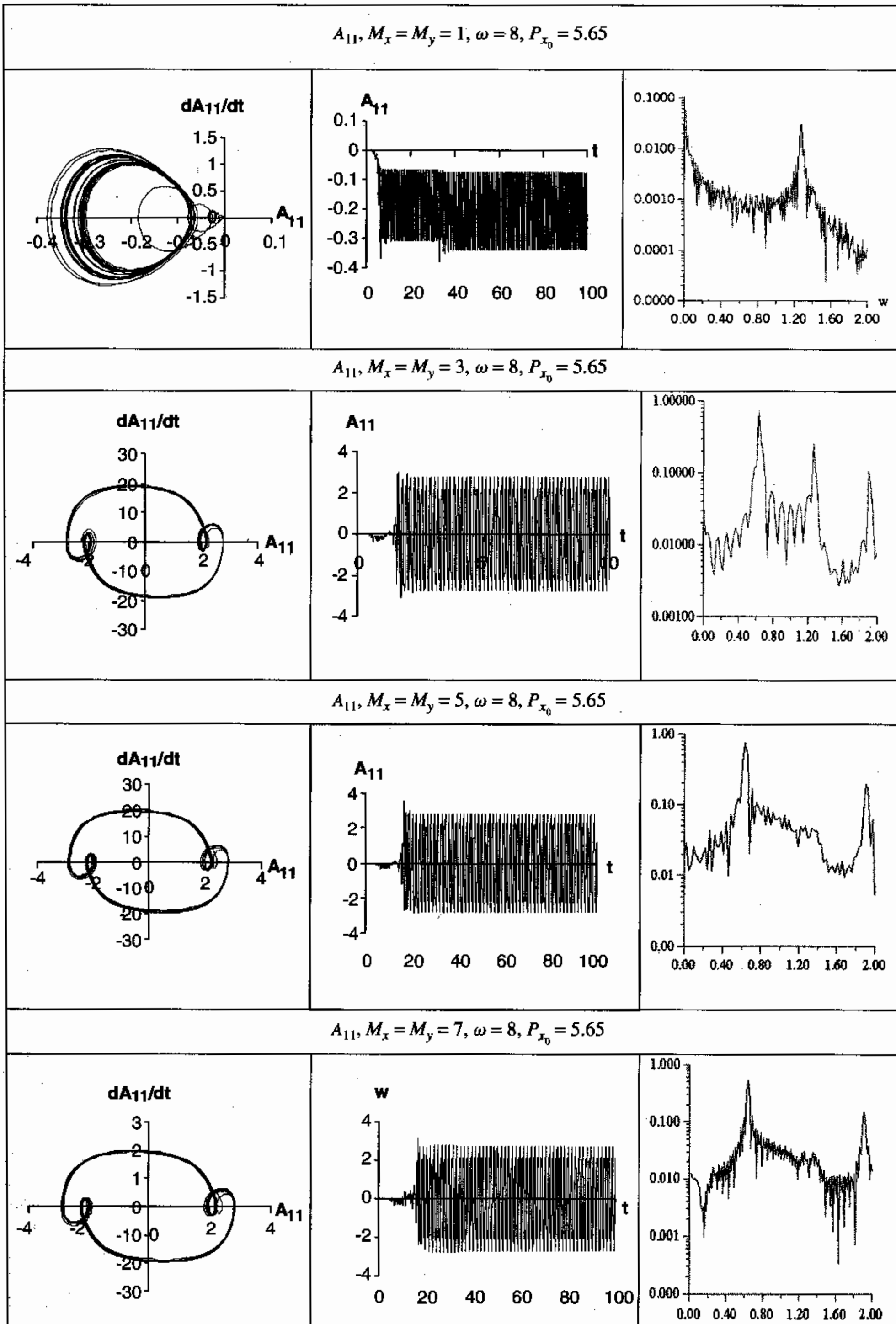

Figure 4. Phase portraits, time histories and power spectra for the attached parameters.

REFERENCES

Awrejcewicz, J. & Krysko, A. V. 2003. *Nonclassical Thermoelastic Problems in Nonlinear Dynamics of Shells*. Berlin, Springer-Verlag.

Krysko, V. A. & Kutsemako, A. N. 1999. *Stability and Vibrations of Non-Homogeneous Shells*. Saratov, Saratov University Press, in Russian.

L'vov, V. S., Predtechenskiy, A. A. & Chernykh, A. I. 1981. Bifurcations and chaos in the system with Taylor's vortices: Real and numerical experiments. *Izviestia VUZov, Radiophysics* 80(3): 1099–1121, in Russian.

Swinney, H. L. & Gollub, J. P. 1981. *Hydrodynamic Instabilities and the Transition to Turbulence*. Berlin, Springer-Verlag.

Waller, I. & Kapral, R. 1984. Synchronization and chaos in coupled nonlinear oscillators. *Phys. Lett. A* 105(4/5): 163–168.